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FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS
ANKARA / TÜRKİYE

INTERNATIONAL GEOMETRY SYMPOSIUM IN MEMORY OF PROF. ERDOĞAN ESİN'S

PROCEEDING BOOK

Online
9 - 10 February 2023

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International Geometry Symposium in Memory of Prof. Erdogan Esin (IGSM Erdogan Esin)

09-10 February 2023, Ankara

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Assoc. Prof. Hülya KADIOĞLU (Prof. Erdogan ESIN's MSc and PhD student), Yıldız Technical University

Preface

My master thesis advisor, Prof. Erdoğan ESİN is dead May 16, 2022. In our department, we organized an international geometry symposium on the Zoom application on February 9-10, 2023 in his memory.

Unfortunately, an earthquake with a magnitude of 7.7 occurred at 04:17 on February 6, 2023, in the Pazarcık district of Kahramanmaraş in our country three days before the symposium. . Earthquake; It was felt intensely in the surrounding provinces, especially in Kahramanmaraş, Hatay, Osmaniye, Adıyaman, Gaziantep, Şanlıurfa, Diyarbakır, Malatya and Adana. I wish God's mercy on our citizens who died in the earthquake and wish a speedy recovery to the injured people. Our pain as a country is great.

Despite all our angles, we held the symposium on 9-10 February 2023. Geometryists who attended the symposium from abroad and from home shared our pain and expressed their condolences. I would like to thank all the participants who supported us by attending the symposium.

Our symposium invited speaker is Prof. Ryszard DESZCZ wanted to talk at the symposium about his student Şahnur YAPRAK, for whom he was the second advisor for her doctoral thesis. So, we also included Assoc. Dr. Şahnur YAPRAK's life in the symposium. We lost her in a tragic traffic accident on the way back from the symposium in June 1996.

Since the online publication of the symposium proceedings book of the papers presented at the symposium coincides with the 2nd century of the Republic of Türkiye, I commemorate with respect and gratitude all our Martyrs and Veterans who fought in the War of Independence, especially Mustafa Kemal ATATÜRK.

I would like to thank Lecturer Dr. Anıl ALTINKAYA, who contributed to the preparation of the symposium proceedings book.

The Symposium Organizing Committee would like to thank very much to our Dean Prof. Suat KIYAK and our Rector Prof. Musa YILDIZ for their support at the symposium.

I hope it will be useful to the world of geometry.

Prof. Aysel TURGUT VANLI

Chairperson of the Symposium Organizing Committee

Contents

Dear Rector

Dear Dean

Distinguished invited speakers and dear participants

I greet you with respect.

Unfortunately, an earthquake with a magnitude of 7.7 occurred at 04:17 on February 6, 2023, in the Pazarcık district of Kahramanmaraş in our country.

Earthquake; It was felt intensely in the surrounding provinces, especially in Kahramanmaraş, Hatay, Osmaniye, Adıyaman, Gaziantep, Şanlıurfa, Diyarbakır, Malatya and Adana. I wish God's mercy on our citizens who died in the earthquake and wish a speedy recovery to the injured people. Our pain as a country is great.

Thank you very much for your support with your participation in the symposium.

You gave honored us.

The symposium will last for two days. In the symposium,10 (ten) geometers from 9 (nine) different countries will make presentations as valuable invited speakers. In addition, there will be 6 (six) sessions in the symposium program and a total of 30 (thirty) presentations will be made. 96 people will participate in the symposium. I believe that the symposium will be beneficial to the scientific world. We organized a symposium in memory of Prof. Erdoğan ESİN.

I want to share my feelings about him. He was my master's thesis advisor. My master's thesis position was determined by my teacher as tangent bundles. I am grateful to my teacher for helping me with my thesis. That's why I am very honored to organize this symposium in memory of Erdoğan ESİN.

He was an idealist scientist who was very meticulous in his scientific studies, examining and questioning everything down to the last detail. Due to the excessive rigor of scientific studies, he published few articles.

However, besides an idealistic humanity, he teaches very well and has very good lecture notes. He used to think of printing these lecture notes as a book, but he couldn't do it.

While our teacher was lecturing, he used to convey his experiences about life beyond mathematics to the students. For this reason, we learned about life in the lessons.

He was a brilliant person. He loved scientific discussions and presented them to academics from different perspectives.

Professor Erdoğan ESİN has given mathematics lessons to thousands of students. He brought 7 scientists to the world of geometry by having 7 master's and 5 doctoral theses in his annual academic life.

We commemorate our teacher and we give our condolences again to his grieving family.

God bless his soul.

I wish you a good symposium.

Kind regards

Organizing Committee Chairperson
Prof. Aysel TURGUT VANLI

Dear Rector

Dear Dean

Dear guests and invited speakers

I greet you with respect.

Welcome to International Geometry Symposium in Memory of Prof. Erdogan ESİN.

Professor Erdoğan ESİN has worked in our department for 30 years and made a great contribution to the development of our department. He contributed to educational, academical and research activities by raising both undergraduate and graduate students in various universities around Türkiye. My condolences to the grieving family of our valuable professor. May his soul rest in peace.

Also, due to the earthquake disaster that happened on 6th of February, we are in mourning and we wish for the best for all earthquake victims who got hurt or affected in any way.

I hope the symposium to be successful and efficient and I present my thanks to everyone involved in making the symposium a reality including our rectorship, faculty of science dean's office and the organising committee of our department.

Sincerely

Head of Department of Mathematics
Prof. A. Duran TÜRKOĞLU

Dear Rector

Dear invited speakers and participants

Warm greetings from Türkiye

Sir, I wish God's mercy to our citizens who lost their lives in the earthquake that took place in our country on February 6, 2023. Unfortunately, we have thousands of wounded. I wish them a speedy recovery as well.

Gazi University is a research university and is also one of the largest universities in Türkiye. The mission of research universities is to conduct pioneering studies in the field of fundamental sciences. Gazi University Department of Mathematics is at the forefront of scientific studies in our country. Hence, it has been making significant contributions both within the country and globally that highlight our university's mission of research. Today, I am sharing the excitement of holding the first international geometry symposium by the mathematics department of our faculty. Thank you for participating in the symposium. This symposium will be the initial step in the opening of the Geometry Department abroad. I believe that the symposium will bring great value to geometry with the participant scientists from all over the world.In the honor of Prof. Erdoğan ESİN , I would like to share brief introductory information about.

Prof. Erdoğan ESİN had been in our mathematics department for 30 years that also served in administration of our faculty.

Vice-chairman of the department from 1997 to 2000, Head of the mathematics department from 2000 to 2003, and served as a member of the faculty board between 1992-1993.

He teached in the mathematics department for many years. Today, there are two lecturers and two research assistants in the Geometry Department. Our two professors working in the Geometry Department are students of Prof. Erdoğan ESİN. For this reason, we dedicated the symposium held today to the memory of our teacher.

I wish God's blessings on our teacher and thank him for his contribution to our faculty.

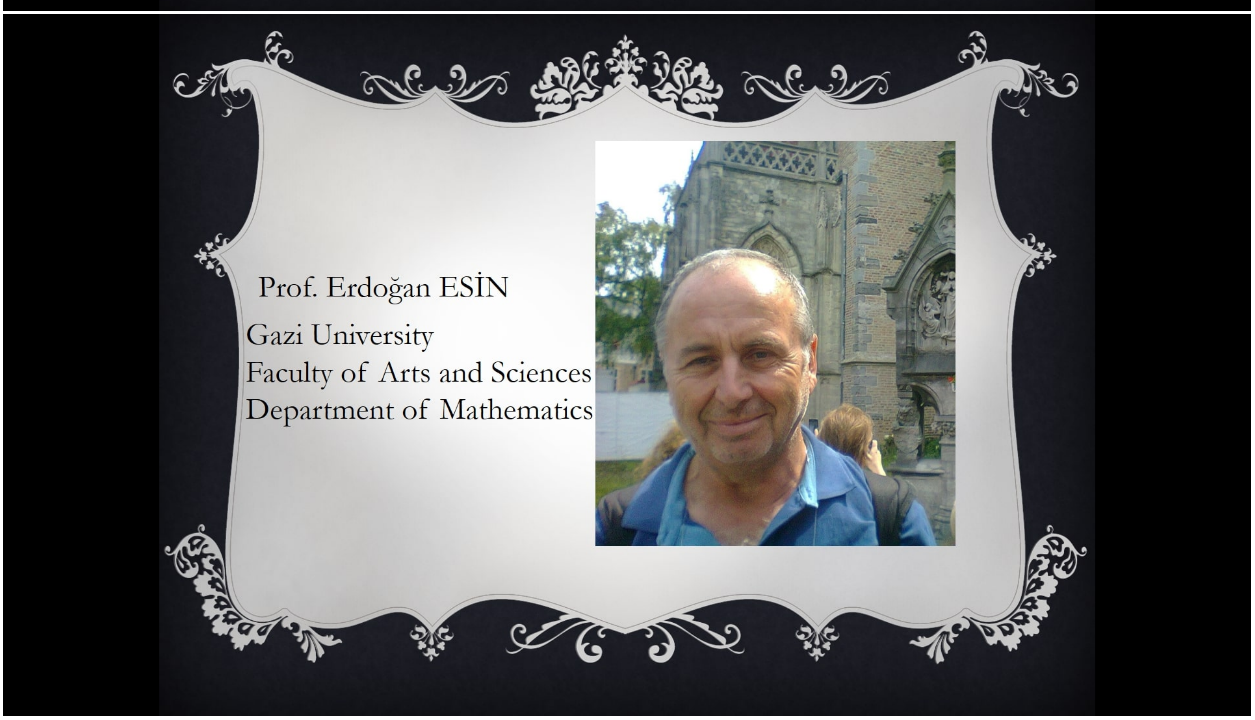
I believe that the symposium will be successful.

I would like to thank Prof. Aysel TURGUT VANLI Chairman of the Organizing Committee, and all the other organizers who contributed to this nice organization.

I respectfully greet the participants.

Prof. Suat KIYAK
Dean of Faculty of Science

Erdoğan ESİN's Life



Prof. Erdoğan ESİN

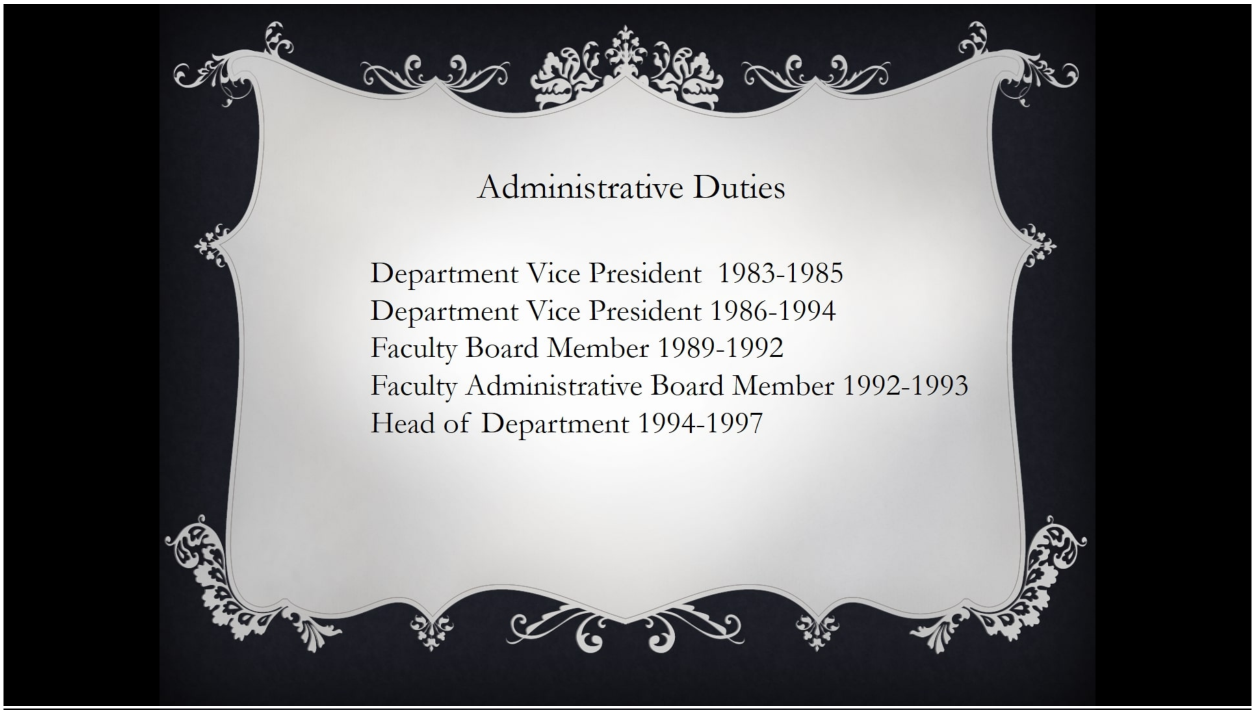
1954-2022
Place of Birth: Ankara
Father of 2
Hobbies: Music, Traveling, Barbeque, Crossword Puzzles

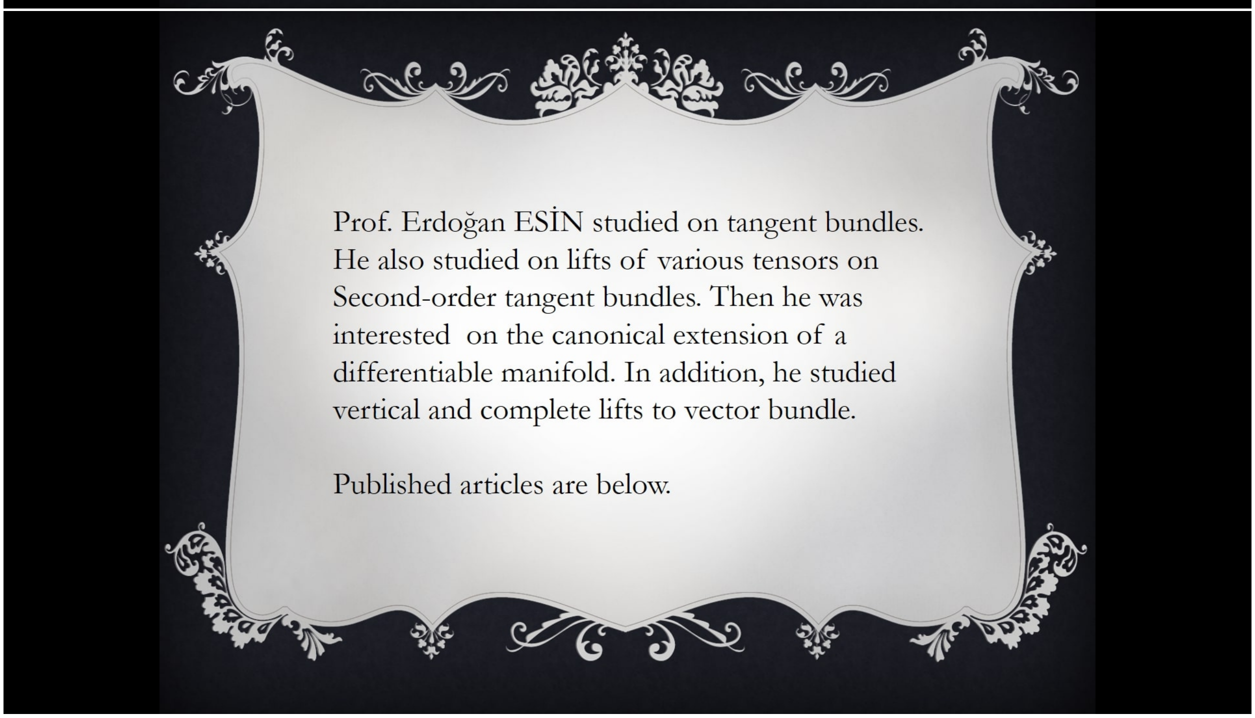
Education and Career

Undergraduate: Ankara University, Department of Mathematics, 1977.
Master's: Ankara University, Department of Mathematics, 1979. Algebra-Geometry (Thesis: Laguerre Curves).
PhD: Ankara University, Department of Mathematics, 1982. Algebra-Geometry (Thesis: Motions Along a Curve and Umbrella Matrices).

Assistant: Ankara University, Department of Mathematics, 1978. Algebra-Geometry.
Asst. Prof. : Gazi University, Department of Mathematics, 1982.
Assoc. Prof. Gazi University, Department of Mathematics, 1987.
Prof. : Gazi University, Department of Mathematics, 1993.

He retired from Gazi University, Department of Mathematics in 2012.


Administrative Duties
Department Vice President 1983-1985
Department Vice President 1986-1994
Faculty Board Member 1989-1992
Faculty Administrative Board Member 1992-1993
Head of Department 1994-1997


Prof. Erdoğan ESİN studied on tangent bundles. He also studied on lifts of various tensors on Second-order tangent bundles. Then he was interested on the canonical extension of a differentiable manifold. In addition, he studied vertical and complete lifts to vector bundle.
Published articles are below.

Scientific Publications

E. Esin, H.H. Hacısalihoğlu,"Curvature Matrices and Darboux Matrices of Motions Along a Curve "Commun, Fac. Sci. Univ. Ank. Series A1, V.35 pp-35-44, 1986
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E. Esin, H.H. Hacısalihoğlu, "Umbrella Matrices and Higher Curvatures of a Curve", Commun. Fac. Sci. Univ. Ank. Series A1, V.35, pp. 27-34, 1986.

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E. Esin, "On Caustic Hypersurfaces In (n+1)-Dimensional Euclidean Space En+1" Journal of the Institute of Science and Tecnology of Gazi Univ.,V.1,No:1,pp.25-27, 1988.

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E. Esin, Ş. Civelek, "İkinci Mertebeden Tanjant Demet Üzerindeki Lift'ler",Gazi Üniversitesi Fen-Edebiyat Fakültesi Matematik-İstatistik Dergisi 2, 117-135, 1989

E. Esin, "İkinci Mertebeden Genişletilmiş Manifoldlara Lift'ler Üzerine", Gazi Univ. Fen-Edebiyat Fak. Mat.-İst. Dergisi, 2, 137-152, 1989.

A. Turgut, E. Esin, "Involute-Evolute Curve Couples of Higher Order in $\mathbb{R}^n$ and Their Horizontal Lifts in $T\mathbb{R}^n$ ", Commun. Fac. Sci. Univ. Ank. Series A1, Math. Statist. V.41, No. 1-2, 125-130, 1992.

A. Turgut, E. Esin,"Some Properties of the Connections on the Tangent Manifold", Gazi Univ.Fen-Edb.Fak.Fen Bil.Dergisi C.7, 125-132, 1997

M. Yıldırım, E. Esin, "On the Canonical Extension of a Differentiable Manifold", Journal of Geometry, 69, 73-78, 2000.

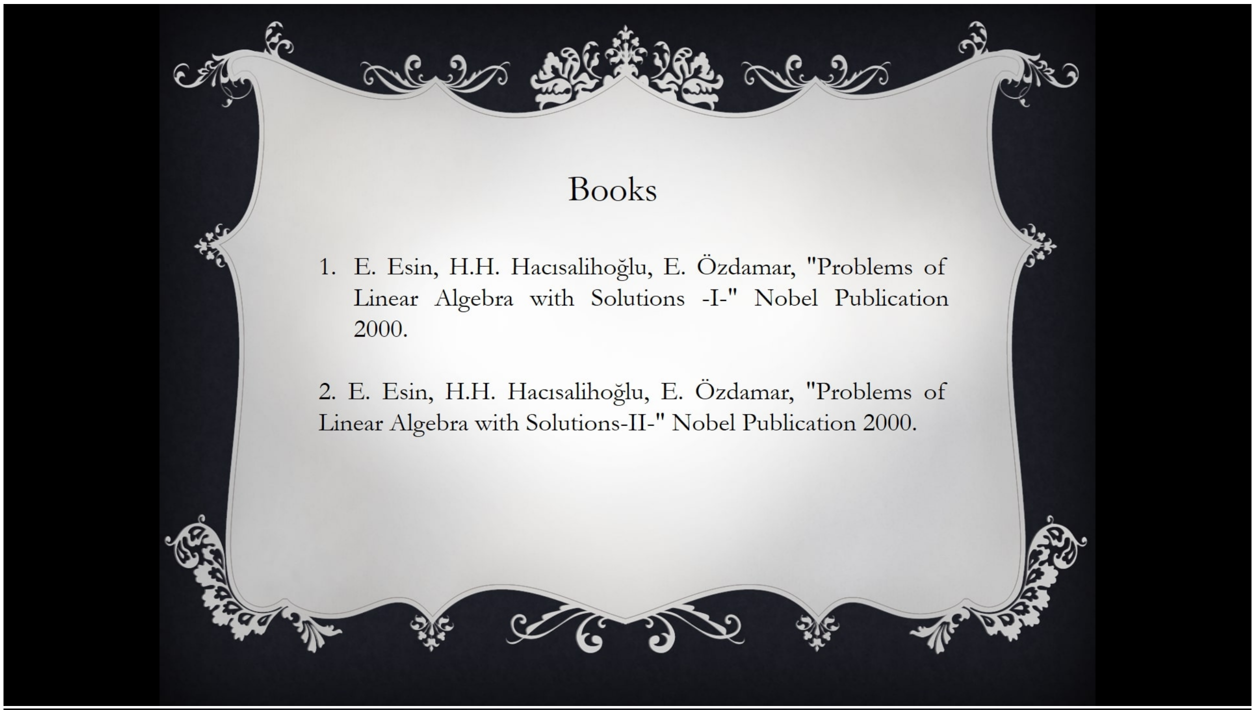
Yildirim, Mehmet; Esin, Erdogan Vertical and complete lifts to vectorbundle.Algebras Groups Geom.24(2007),no. 1,75–84.

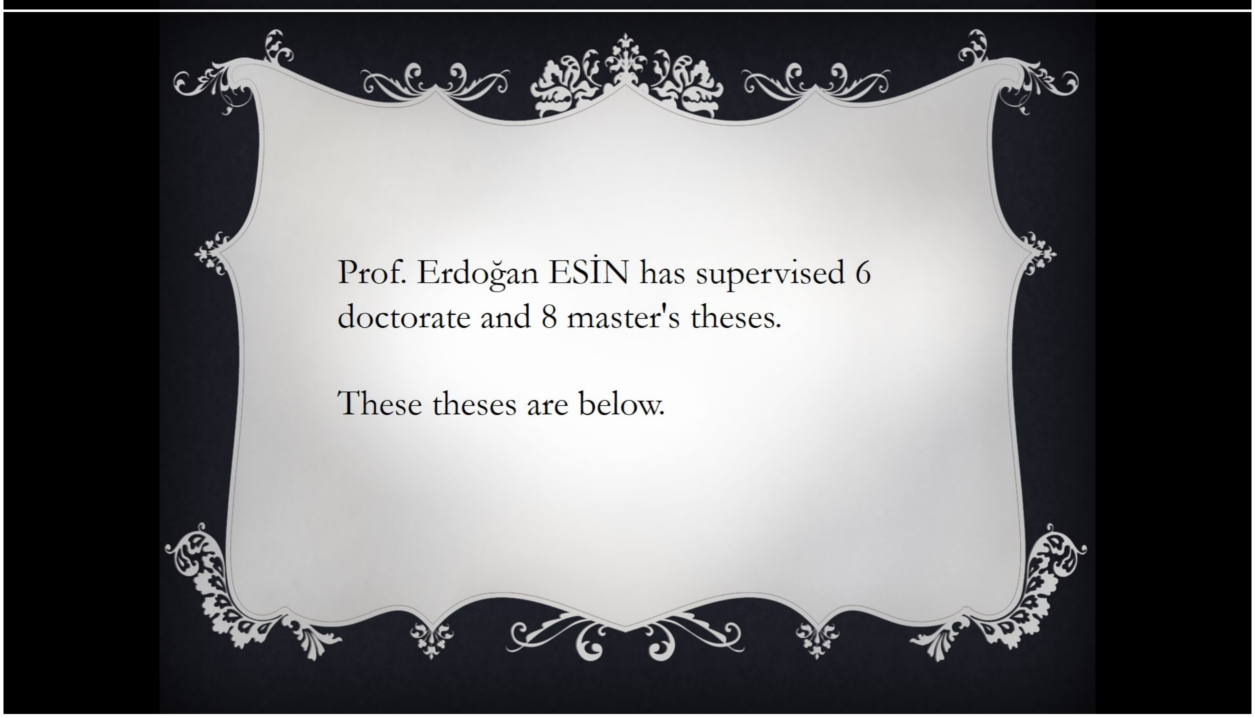
Ozkan, M.; Esin, E. Vertical lifts of tensor fields to vector bundle.Algebras Groups Geom.25(2008),no. 1,21–29.

Kadioglu, Hulya; Esin, Erdogan; Yayli, Yusuf On the prolongations of representations of Lie algebras.Hadronic J.33(2010),no. 6,623–636

Kadioğlu, Hülya; Esin, Erdoğan; Yayli, Yusuf Prolongations of Lie algebra representations. Adv. Appl. Math. Sci.10(2011),no. 5,533–542.

Çiçek Gözütok, Ayşe; Esin, Erdoğan Tangent bundle of hypersurface with semi-symmetric metric connection.Int. J. Contemp. Math. Sci.7(2012),no. 5-8,279–289


Books
1. E. Esin, H.H. Hacısalihoğlu, E. Özdamar, "Problems of Linear Algebra with Solutions -I-" Nobel Publication 2000.
2. E. Esin, H.H. Hacısalihoğlu, E. Özdamar, "Problems of Linear Algebra with Solutions-II-" Nobel Publication 2000.


Prof. Erdoğan ESİN has supervised 6 doctorate and 8 master's theses.
These theses are below.


Supervised Theses
Şevket Civelek, (Master's Thesis) "Lifts on Manifolds Extended from the Second Order", Gazi Univ. Science. Ens., 1988.
Aysel Turgut, (Master's Thesis) Gazi Univ. Science. Ens., 1989.
Şevket Civelek, (PhD Thesis) "High Order Lifts to Extended Vector Bundles", Gazi Univ. Science. Ens., 1993.
Mehmet Yıldırım, (Master's Thesis) "Differential Geometry of Manifolds Extended from Second Order" Gazi Univ. Science. Ens., 1997.


Figen Çilingir (Master's Thesis) Prolongation Of Hypersurfaces To Tangent Bundles , 1997.
Mustafa Özkan, (Master's Thesis) Gazi Univ. Science. Ens., 1998.
Mehmet Yıldırım, (PhD Thesis) "Manifold Extensions and Lifts to Vector D e Texts", Gazi Univ. Science. Ens., 2003.
Mustafa Özkan, (PhD Thesis) Prolongations Of Hypersurfaces To Vector Bundles, Gazi Univ. Science. Ens., 2006.

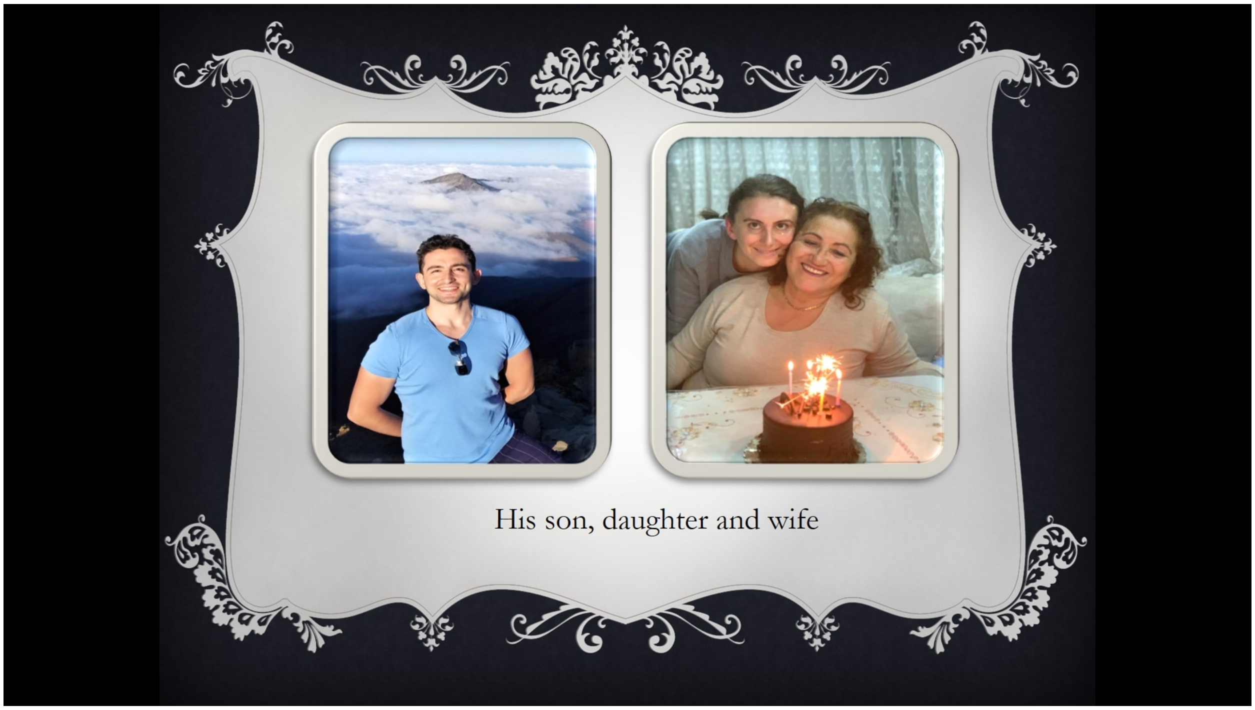
Hülya Kaya, (Master's Thesis) "Transportation of G-structures to Tangent Bundles", Gazi Univ. Science. Ens., 2006.

Osman Keçilioğlu (Ph. D. Thesis) Prolongations Of Hypersurfaces To Tangent Bundles Of Order Two, 2010

Hülya (Kaya) Kadıoğlu, (Ph. D. Thesis) Prolongations Of Representations Of Lie Groups To Tangent Bundles, 2011

Ayşe Çiçek Gözütok (Ph. D. Thesis), Invariant Hypersurfaces With Semi-symmeytric Metric Connection Of F_α (k,1)-structure Manifold, 2012.

Erdoğan and his wife


His son, daughter and wife


GAZİ ÜNİVERSİTESİ
Gazi
1926
Sayın Prof. Dr. Erdoğan ESİN
Aralık-1982-Eylül-2013 tarihleri arasında
Gazi Üniversitesi Fen Fakültesi Matematik
Bölümünde yapmış olduğunuz başarılı
hizmetlerinizden dolayı
bölüm olarak teşşekür eder,
sağlıklı emeklilikler dileriz
Matematik Bölümü
27.02.2014
Retirement plaque


May God have mercy on him,
I hope his place is heaven
God rest the soul
Our condolences again to his grieving family.
we remember him fondly

Proceedings of the International Geometry
Symposium in Memory of Prof. Erdoğan ESİN
9-10 February 2023

CAUCHY-RIEMANN GEOMETRY[1] :AN INTRODUCTION TO THE MAIN PROBLEMS

Elisabetta BARLETTA [2], Sorin DRAGOMİR[3] and Francesco ESPOSITO[4]

ABSTRACT

The ordinary Cauchy-Riemann system $\overline{\partial} f = 0$ on $\mathbb{C}^n$ ($n \geq 2$) induces on every real hypersurface $M \subset \mathbb{C}^n$ the *tangential Cauchy-Riemann equations*

$$\overline{\partial}_M u = 0 \tag{1}$$

(a first order overdetermined PDE system, with variable C^∞ coefficients) and every C^1 solution $u : M \to \mathbb{C}$ to (1) is a *CR function* on M. A CR structure is a recast of (1) as an involutive complex distribution $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$, of complex rank $n-1$, and the restriction to M of a holomorphic function (on a neighborhood $\Omega \supset M$) is a solution to (1). The *CR extension* problem is whether a point $x_0 \in M$ admits a neighborhood $\Omega \subset \mathbb{C}^n$ such that the restriction morphism $\mathcal{O}(\Omega) \to \mathrm{CR}^1(U)$ is an epimorphism (with $U = \Omega \cap M$). Given an *abstract* CR structure $T_{1,0}(M)$ on a real $(2n-1)$-dimensional manifold (not necessarily embedded into $\mathbb{C}^n$) the *CR embedding* problem is whether a point $x_0 \in M$ admits a neighborhood $U \subset M$ and a CR immersion $\Psi : U \to \mathbb{C}^n$ [so that the portion of $T_{1,0}(M)$ over U is actually induced by the complex structure of the ambient space $\mathbb{C}^n$]. We review results (old and new) on the two fundamental problems mentioned above, with an emphasis on the differential geometric objects needed in their study (cf. [3] and [1]), and indicate their relationship to mathematical physics (cf. [7], [8], [5], and [6]).

[1]Lecture delivered at the *International Geometry Symposium in Memory of Professor Erdogan Esin*, Gazi University, Faculty of Science, 9-10 February 2023.

[2]Università degli Studi della Basilicata, Dipartimento di Matematica, Informatica ed Economia, Potenza, Italy, e-mail: `elisabetta.barletta@unibas.it`

[3]Università degli Studi della Basilicata, Scuola di Ingegneria, Potenza, Italy, e-mail: `sorin.dragomir@unibas.it`

[4]Università degli Studi della Basilicata, Dipartimento di Matematica, Informatica ed Economia, Potenza, Italy, e-mail: `f.esposito@unibas.it`

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[1] E. Barletta & S. Dragomir & K.L. Duggal, *Foliations in Cauchy-Riemann geometry*, Mathematical Surveys and Monographs, Vol. 140, American Mathematical Society, 2007.

[2] *Selected topics in Cauchy-Riemann Geometry*, Edited by S. Dragomir, Quaderni di Matematica, Napoli, Vol. 9, 2001.

[3] S. Dragomir & G. Tomassini, *Differential Geometry and Analysis on CR Manifolds*, Progress in Mathematics, Vol. 246, Birkhäuser, Boston-Basel-Berlin, 2006.

[4] *Geometry of Cauchy-Riemann submanifolds*, Edited by S. Dragomir & M.H. Shahid & F.R. Al-Solamy, Springer Science, Singapore, 2016.

[5] C.D. Hill & J. Lewandowski & P. Nurowski, *Einstein's equations and the embedding of 3-dimensional CR manifolds*, `arXiv:0709.3660v2 [math.DG] 22 May 2008`

[6] R. Penrose, *Physical space-time and nonrealizable CR structures*, Bull. A.M.S., N.S., 8(1983), 427-448.

[7] I. Robinson, *A solution of the Maxwell-Einstein equations*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys., 7(1959), 351-352.

[8] I. Robinson & A. Trautman, *Cauchy-Riemann structures in optical geometry*, 4th Marcel Grossmann Meeting on the Recent Developments of General Relativity, 1985.

After the publication of the abstract of the present lecture, its objectives appeared as too ambitious for a fifty minutes talk, and the Authors chose to confine the presentation to the CR embedding problem, and relegate other fundamental questions (such as the CR extension problem, or the applications of CR geometry and analysis to theoretical physics) to a further seminar.

1. CR structures, the way they occur

Let M be a real $(2n+k)$-dimensional C^∞ manifold, with $n \geq 1$ and $k \geq 1$.

Definition 1. A complex subbundle $\mathcal{H} \subset T(M) \otimes \mathbb{C}$, of the complexified tangent bundle, of complex rank n, is called an *almost CR structure*[5] if

$$\mathcal{H}_x \cap \overline{\mathcal{H}}_x = \{0\}, \quad x \in M. \tag{2}$$

The integer n is the *CR dimension*, while k is the *CR codimension*. The pair $(n,\, k)$ is the *type* of the given almost CR structure. An almost CR structure $\mathcal{H}$ is *formally integrable* if for every open set $U \subset M$

$$Z,\, W \in C^\infty(U,\, \mathcal{H}) \Longrightarrow \big[Z,\, W\big] \in C^\infty\big(U,\, \mathcal{H}\big). \tag{3}$$

A *CR structure* is a formally integrable almost CR structure. A pair $(M,\, \mathcal{H})$ consisting of a manifold M and an (almost) CR structure $\mathcal{H}$ is an (*almost*) *CR manifold*. Let $(M,\, \mathcal{H})$ be an almost CR manifold, of type $(n,\, k)$. Let us consider the first order differential operator

$$\overline{\partial}_M : C^1\big(M,\, \mathbb{C}\big) \to C\big(\overline{\mathcal{H}}^*\big),$$

$$\big(\overline{\partial}_M u\big)\overline{Z} = \overline{Z}(u), \quad u \in C^1(M, \mathbb{C}), \quad Z \in C^\infty(\mathcal{H}).$$

[5]Key Words: *Tangential CR equations, CR function, Tanaka-Webster connection, sublaplacian, Fefferman metric, Robinson-Trautman construction.*

Definition 2. $\overline{\partial}_M$ is the *tangential Cauchy-Riemann operator* and

$$\overline{\partial}_M u = 0 \tag{4}$$

are the *tangential C-R equations*. A C^1 solution u to (4) is a *CR function*.

Let us look at a few examples of (almost) CR structures.

1.1 Real hypersurfaces in $\mathbb{C}^{n+1}$. Let $M \subset \mathbb{C}^{n+1}$ be a real hypersurface, and let us set

$$T_{1,0}(M)_x := \big[T_x(M) \otimes_{\mathbb{R}} \mathbb{C}\big] \cap T'\big(\mathbb{C}^{n+1}\big)_x, \quad x \in M,$$

where $T'(\mathbb{C}^{n+1})$ is the holomorphic tangent bundle over $\mathbb{C}^{n+1}$. Then $T_{1,0}(M)$ is a CR structure on M, of type $(n,\, 1)$. In particular, if M is given by a *defining function* $\rho \in C^\infty\big(\Omega,\, \mathbb{R}\big)$, with $\Omega \subset \mathbb{C}^{n+1}$ open, i.e.

$$M = \{x \in \Omega \,:\, \rho(x) = 0\}, \quad D\rho(x) \neq 0, \quad x \in M,$$

then the tangential C-R equations on M are

$$\overline{L}_\alpha u = 0, \quad 1 \le \alpha \le n,$$
$$L_\alpha \equiv a^j_\alpha \frac{\partial}{\partial z^j}\,, \quad a^j_\alpha \frac{\partial \rho}{\partial z^j} = 0.$$

Let

$$\Omega_{n+1} = \Big\{(z, \zeta) \in \mathbb{C}^n \times \mathbb{C} \,:\, \frac{1}{2i}\big(\zeta - \overline{\zeta}\big) > |z|^2\,\Big\},$$
$$z = \big(z^1\,, \cdots, z^n\big), \quad |z|^2 = \delta_{\alpha\beta}\, z^\alpha\, \overline{z}^\beta\,,$$

be the *Siegel domain*, or *generalized upper half plane*. Its boundary $\partial\Omega_{n+1}$ is a real hypersurface in $\mathbb{C}^{n+1}$, and hence carries the induced CR structure

$$T_{1,0}\big(\partial\Omega_{n+1}\big) \subset T\big(\partial\Omega_{n+1}\big) \otimes \mathbb{C}^{n+1}.$$

$T_{1,0}\big(\partial\Omega_{n+1}\big)$ is the span of $\{L_\alpha \,:\, 1 \le \alpha \le n\}$ where

$$L_\alpha \equiv \frac{\partial}{\partial z^\alpha} + 2\, i\, \overline{z}^\alpha \frac{\partial}{\partial \zeta}\,,$$

hence the tangential C-R equations (on the boundary of the Siegel domain) are

$$\frac{\partial u}{\partial \overline{z}^\alpha} - 2\, i\, z^\alpha \frac{\partial u}{\partial \overline{\zeta}} = 0, \quad 1 \le \alpha \le n.$$

This is precisely the context in which the tangential C-R equations were discovered by H. Lewy[6], though confined to the case $n = 1$ i.e. on the boundary of the Siegel domain $\Omega_2 \subset \mathbb{C}^2$. H. Lewy started with the Dirichlet problem for the (ordinary) Cauchy-Riemann equations on the Siegel domain

$$\frac{\partial F}{\partial \overline{z}} = 0, \quad \frac{\partial F}{\partial \overline{\zeta}} = 0 \quad \text{in } \ \Omega_2\,, \tag{5}$$
$$F = u \quad \text{on } \ \partial\Omega_2\,, \tag{6}$$

with the boundary datum $u \in C^\infty\big(\partial\Omega_2\,,\, \mathbb{C}\big)$, and posed the problem of the existence of a solution $F \in \mathcal{O}\big(\Omega_2\big) \cap C^\infty\big(\overline{\Omega}_2\big)$ [to (5)-(6)] i.e. F is holomorphic in Ω_2 and C^∞ up to the boundary. It is

[6] Cf. H. Lewy, *On the local character of the solutions of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex variables*, Ann. Math., 64(1956), 514-522.

an elementary matter that a necessary condition for such a solution F to exist, is that its boundary values u satisfies the tangential C-R equations on $\partial\Omega_2$.

Let $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$ be the *Heisenberg group* i.e. the non commutative Lie group $\mathbb{R}^{2n+1} \approx \mathbb{C}^n \times \mathbb{R}$ with the group law

$$(z,\, t) \cdot (w,\, s) = \big(z + w,\; t + s + 2\,\mathrm{Im}(z \cdot \overline{w})\big), \quad z \cdot \overline{w} = \sum_{\alpha=1}^{n} z^\alpha\, \overline{w}^\alpha \,.$$

Let us consider the (left invariant) complex vector fields

$$L_\alpha \equiv \frac{\partial}{\partial z^\alpha} + i\, \overline{z}^\alpha\, \frac{\partial}{\partial t}\,, \quad 1 \le \alpha \le n,$$

and let $\mathcal{H}$ be the span of the L_α's. Then $\mathcal{H}$ is a CR structure, of type $(n,\, 1)$, on $\mathbb{H}_n$, and the map

$$\phi : \mathbb{H}_n \to \partial\Omega_{n+1}\,,$$
$$\phi(z,\, t) = \big(z,\, t - i\,|z|^2\big), \quad z \in \mathbb{C}^n\,, \quad t \in \mathbb{R},$$

is a *CR isomorphism* i.e. a C^∞ diffeomorphism preserving the CR structures

$$(d_x\phi)\mathcal{H}_x \subset T_{1,0}\big(\partial\Omega_{n+1}\big)_{\phi(x)}\,, \quad x \in \mathbb{H}_n\,.$$

When $n = 1$ the differential operator

$$\overline{L}_1 \equiv \frac{\partial}{\partial \overline{z}} - i\, z\, \frac{\partial}{\partial t} \in C^\infty\big(T(\mathbb{H}_n) \otimes \mathbb{C}\big)$$

is the *Lewy operator*.

The Lewy operator is known[7] to be *unsolvable* i.e. there exist functions $f \in C^\infty\big(\mathbb{H}_n\big)$ such that the equation $\overline{L}_1 u = f$ possesses no C^∞ solution u. The unsolvability of the Lewy operator relates Cauchy-Riemann analysis to an important chapter[8] of the theory of PDEs, and prompted the development of the latter. We shall come back again on the significance of the unsolvability property of $\overline{L}_1$, as related to the CR embeddability problem.

1.2 Orbit spaces of null Killing vector fields. Let $\mathfrak{M}$ be a real $(2n+2)$-dimensional C^∞ manifold, equipped with the Lorentzian metric g, and let $N \in \mathfrak{X}(\mathfrak{M})$ be a null [i.e. $g(N,\, N) = 0$, $N_x \neq 0$ for any $x \in \mathfrak{M}$] Killing (i.e. $\mathcal{L}_N g = 0$) tangent vector field. Let W and C be the Weyl and Cotton tensor fields i.e.

$$W_{ijk\ell} = R_{ijk\ell} - L_{jk}\, g_{i\ell} - L_{i\ell}\, g_{jk} + L_{j\ell}\, g_{ik} + L_{ik}\, g_{j\ell}\,,$$
$$C_{jk\ell} = \nabla_\ell L_{jk} - \nabla_k L_{j\ell}\,,$$

where

$$L_{jk} = \frac{1}{2n}\Big\{R_{jk} - \frac{R}{2(2n+1)}\, g_{jk}\Big\}\,,$$

and let us assume that

$$N \rfloor W = 0, \quad N \rfloor C = 0.$$

Next, let us consider the differential 1-forms and tangent vector field

$$\theta,\ \sigma \in \Omega^1(\mathfrak{M}), \quad V \in \mathfrak{X}(\mathfrak{M}),$$

[7]Cf. H. Lewy, *An example of a smooth linear partial differential equation without solution*, Ann. Math., 66(1957), 155-158.
[8]Cf. L. Hörmander, *Differential equations without solutions*, Mathematische Annalen, 140(1960), 169-173. Cf. also V.V. Grusin, *A certain example of a differential equation without solutions*, Math. Notes, 10(1971), 499-501.

$$\theta(X) = g(X,\, N), \quad \sigma(X) = L(X,\, N), \quad g(V,\, X) = \sigma(X).$$

Also, let us consider the field of endomorphisms

$$J : T(\mathfrak{M}) \to T(\mathfrak{M}), \quad JX = \nabla_X N.$$

Then $(J,\, N,\, V,\, \theta,\, \sigma)$ is an f-structure on $\mathfrak{M}$, with two complmented frames, in the sense of D.E. Blair et al.[9], compatible to the Lorentzian metric g i.e.

$$g(JX,\, JY) = g(X,\, Y) - \theta(X)\sigma(Y) - \sigma(X)\,\theta(Y).$$

If $H = \operatorname{Ker}(\theta) \cap \operatorname{Ker}(\sigma)$ then J descends to a complex structure $J : H \to H$ and $\operatorname{Spec}\big(J^{\mathbb{C}}\big) = \{\pm i\}$. The eigenbundle

$$H^{1,0} = \operatorname{Eigen}\big(J^{\mathbb{C}}\,;\, +i\big) \subset T(\mathfrak{M}) \otimes \mathbb{C}$$

is an almost CR structure of type $(n,\, 2)$ on $\mathfrak{M}$, which in general isn't formally integrable[10]. Let $M = \mathfrak{M}/N$ be the orbit space, consisting of the maximal integral curves of N, and let us assume that M is a C^∞ manifold[11]. Let $\mathcal{F}$ be the foliation of $\mathfrak{M}$ by orbits of N, so that $M = \mathfrak{M}/N$ is the leaf space [of the foliated manifold $(\mathfrak{M},\, \mathcal{F})$]. Then θ is a basic 1-form[12] $\theta \in \Omega^1_B(\mathcal{F})$ while the Lorentz metric g, the endomorphism J, and the 1-form σ are invariant under sliding along the leaves of $\mathcal{F}$. Consequently[13] the almost CR structure $H^{1,0}$ projects (*via* $\pi : \mathfrak{M} \to M = \mathfrak{M}/N$) on a CR structure $\mathcal{H}$ on M, of type $(n,\, 1)$.

1.3 Bejancu's CR submanifolds. Let M be a submanifold of a Kählerian manifold $(\tilde{M},\, J,\, \tilde{g})$, where J and $\tilde{g}$ are respectively the complex structure and the Kählerian metric on N, and let $g = j^*\tilde{g}$ be the first fundamental form of the given immersion $j : M \hookrightarrow \tilde{M}$. Let $\mathcal{D}$ be a C^∞ distribution on M.

Definition 3. The pair $(M,\, \mathcal{D})$ is said to be a *CR submanifold*[14] of the Kählerian manifold $(\tilde{M},\, J,\, \tilde{g})$ if

i) $\mathcal{D}$ is J-invariant i.e. $J_x\big(\mathcal{D}_x\big) \subset \mathcal{D}_x$, $x \in M$,

ii) The orthogonal complement $\mathcal{D}^\perp$ of $\mathcal{D}$ in $\big(T(M),\, g\big)$ is J-anti-invariant i.e. $J_x\big(\mathcal{D}^\perp_x\big) \subset E(j)_x$, $x \in M$, where $E(j) \to M$ is the normal bundle of the given immersion j.

Let $(M,\, \mathcal{D})$ be a CR submanifold of the Kählerian manifold $(\tilde{M},\, J,\, \tilde{g})$. The complex structure of the ambient space descends to a complex structure $J_M : \mathcal{D} \to \mathcal{D}$. By a result of D.E. Blair & B-Y. Chen[15] the eigenbundle

$$\mathcal{H} = \operatorname{Eigen}\big(J^{\mathbb{C}}_M\,;\, +i\big) \subset T(M) \otimes \mathbb{C}$$

is a CR structure on M, so that $(M,\, \mathcal{H})$ is a CR manifold of type $(n,\, k)$, where

$$\dim_{\mathbb{R}} \mathcal{D}_x = 2n, \quad \dim_{\mathbb{R}} \mathcal{D}^\perp_x = k, \quad x \in M.$$

[9]Cf. D.E. Blair & G.D. Ludden & K. Yano, *Differential Geometric Structures on principal toroidal bundles*, Trans. Amer. Math. Soc., 181(1973), 175-184.

[10]The precise integrability conditions for $H^{1,0}$ were not studied.

[11]This is the case, for instance, when N is regular in the sense of R. Palais, *A global formulation of the Lie theory of transformation groups*, Memoirs of Amer. Math. Soc., No. 22, 1957.

[12]That is $\theta(N) = 0$ and $N \rfloor d\theta = 0$.

[13]By a result of C.R. Graham, *On Sparling's characterization of Fefferman metrics*, American J. Math., 109(1987), 853-874.

[14]CR submanifolds were introduced by A. Bejancu, *CR submanifolds of a Kähler manifold*, I-II, Proc. Amer. Math. Soc., 69(1978), 134-142; Trans. Amer. Math. Soc., 250(1979), 335-345.

[15]Cf. D.E. Blair & B-Y. Chen, *On CR submanifolds of Hermitian manifolds*, Israel J. Math., 34(1979), 353-363.

A program devoted to recovering results in complex analysis, from real hypersurfaces $M \subset \mathbb{C}^{n+1}$ to CR submanifolds of a Hermitian manifold $\tilde{M}$ (perhaps with the ambient space $\tilde{M}$ a Kählerian, or a locally conformal Kähler, manifold) was started by E. Barletta & S. Dragomir[16].

1.4. Contact Riemannian manifolds. Let M be a real $(2n+1)$-dimensional C^∞ manifold. The almost contact structure[17] $(\varphi,\, \xi,\, \eta)$ underlying an almost contact metric structure $(\varphi,\, \xi,\, \eta,\, g)$ on M is *normal* if

$$\big[\varphi\,,\,\varphi\big] + 2(d\eta)\otimes\xi = 0 \tag{7}$$

[and normality may be geometrically interpreted as the integrability condition for the almost complex structure on $M \times \mathbb{R}$, naturally associated to $(\varphi,\, \xi,\, \eta)$]. Let us consider the 2-form $\Phi(X,Y) = g(X,\, \varphi Y)$. The *contact condition* is

$$d\eta = \Phi. \tag{8}$$

A *contact metric structure* is an almost contact metric structure obeying to the contact condition (8) [and $(M,\, (\varphi,\, \xi,\, \eta,\, g)$ is a *contact Riemannian manifold*].

Let $(\varphi,\, \xi,\, \eta)$ be an almost contact structure on M. The restriction J of the endomoprhism φ to $\mathrm{Ker}(\eta)$ is a complex structure along $\mathrm{Ker}(\eta)$ and hence

$$\mathcal{H} = \mathrm{Eigen}\big(J^{\mathbb{C}}\,;\,+i\big) \subset \mathrm{Ker}(\eta)\otimes\mathbb{C}$$

is an almost CR structure on M, of type $(n,\, 1)$. As observed long ago by S. Ianuş[18] normality (7) is a sufficient condition for the integrability of $\mathcal{H}$. The converse is however false in general. The integrability of $\mathcal{H}$ on a contact Riemannian manifold was characterized[19] by the condition $Q = 0$ where Q is the *Tanno tensor* field[20].

We take the occasion to break a lance in favor of the study of the CR geometry of contact Riemannian manifolds with $Q \neq 0$ i.e. in the absence of integrability.

Of course, as $\mathcal{H}$ is only an almost CR structure, S.M. Webster's theory[21] doesn't apply (and basic tools in pseudohermitian geometry, such as the Tanaka-Webster connection, the Fefferman metric, the sublaplacian, aren't *a priori* defined). Nevertheless, as shown by S. Tanno[22] the wealth of additional geometric structure on a given contact Riemannian manifold exhibits strong formal similarities to the geometric structure of a strictly pseudoconvex CR manifold (endowed with a positively oriented contact form) and compensates for the lack of integrability of $\mathcal{H}$, to the point that an analog to the Tanaka-Webster connection (the *Tanno connection*), and of the sublaplacian (Tanno's second order degenerate elliptic operator Δ_H), can be built. For instance

$$\Delta_H f \equiv \Delta f - \xi\big(\xi(f)\big),$$

$$\Delta f \equiv \frac{1}{\sqrt{\mathfrak{g}}}\,\frac{\partial}{\partial x^i}\Big(\sqrt{\mathfrak{g}}\,g^{ij}\,\frac{\partial f}{\partial x^j}\Big), \quad f \in C^2(M),$$

[16] Cf. E. Barletta & S. Dragomir, *CR submanifolds of Hermitian manifolds and the tangential CR equations*, in *Geometry of Cauchy-Riemann submanifolds*, 91-122, Springer, Singapore, 2016.

[17] In the sense of D.E. Blair, *Contact manifolds in Riemannian geometry*, vol. 509, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[18] Cf. S. Ianuş, *Sulle varietà di Cauchy-Riemann*, Rend. dell'Accad. Sci. Fis. Mat., Napoli, 39(1972), 191-195.

[19] Cf. E. Barletta & S. Dragomir, *Differential equations on contact Riemannian manifolds*, Annali della Scuola Normale Superiore di Pisa, Cl. Sci., Ser. IV, (1)XXX(2001), 63-96.

[20] Cf. S. Tanno, *Variational problems on contact Riemannian manifolds*, Trans. Amer. Math. Soc., (1)314(1989), 349-379.

[21] S.M. Webster, *Pseudohermitian structures on a real hypersurface*, J. Diff. Geometry, 13(1978), 25-41.

[22] Cf. *op. cit.*

[Δ is the Laplace-Beltrami operator of the Riemannian manifold $(M, \, g)$].

As a result, considerations very much like Webster's hold on a contact Riemannian manifold, sort of in the presence of the "ghost" of pseudohermitian geometry. Tanno's pioneering study was carried further by E. Barletta & S. Dragomir[23], who proved the subellipticity (of order $\epsilon = 1/2$) of Tanno's sublaplacian Δ_H i.e. Δ_H is formally self-adjoint and[24]

$$\forall \, x \in M, \;\; \exists \, U \subset M \;\; \text{open}, \quad x \in U, \quad \exists \, C > 0,$$
$$\forall \, f \in C^\infty_0(U) \;\; : \;\; \|f\|^2_\epsilon \leq C \Big\{ \big| \big(\Delta_H f \, , \, f \big)_{L^2} \big| + \big\| f \big\|^2_{L^2} \Big\}$$

with $\epsilon = 1/2$, and built a contact Riemannian analog to the Fefferman metric. The same program was further pursued by D.E. Blair & S. Dragomir[25]. A problem left open there[26] (that is whether the scalar curvature of the contact Riemannian analog to the Fefferman metric projects on Tanno's scalar curvature) was recently solved by Masayoshi Nagase[27]. A contact Riemannian analog to the Folland-Stein normal coordinates[28] was devised by S. Dragomir & D. Perrone[29]. A Bochner-type formula leading to an estimate by below of the first nonzero eigenvalue of Tanno's sublaplacian Δ_H (a contact Riemannian analog to the Lichnerowicz-Obata theorem in Riemannian geometry) was obtained by Feifan Wu & Wei Wang[30]. It should be mentioned that the result by F. Wu & W. Wang[31] is legitimated by the subellipticity of Δ_H (as previously established by E. Barletta & S. Dragomir[32]). Indeed, by a result of A. Menikoff & J. Sjöstrand[33] if M is compact then $-\Delta_H$ has a discrete spectrum

$$0 < \lambda_1 < \cdots < \lambda_\nu < \cdots \uparrow +\infty \, .$$

The same Authors[34] solved the (contact Riemannian analog to the) CR Yamabe problem. As well as in their previous work on the Lichnerowicz-Obata theorem, W. Wang & F. Wu focus[35] on the differential geometric part of the problem[36].

Among the open problems in almost CR geometry on a contact Riemannian manifold that in the opinion of the present lecturer, ought to be addressed, is building a (contact Riemannian analog to)

[23] Cf. *op. cit.*

[24] Here $\| \cdot \|_\epsilon$ is the Sobolev norm of order ϵ i.e. $\|u\|^2_\epsilon = \int \big(1 + |\xi|^2\big)^\epsilon \, \big|\hat{u}\big|^2 d\xi$ where $\hat{u}$ is the Fourier transform of u.

[25] Cf. D.E. Blair & S. Dragomir, *Pseudohermitian geometry on contact Riemannian manifolds*, Rendiconti di Matematica, Serie VII, Volume 22, Roma (2002), 275-341.

[26] Cf. *op. cit.*

[27] Cf. M. Nagase, *On the curvature of the Fefferman metric of contact Riemannian manifolds*, Tohoku Math. J., 71(2019), 425-436.

[28] Playing a major role in the (partial) solution to the CR Yamabe problem, cf. D. Jerison & J.M. Lee, *The Yamabe problem on CR manifolds*, J. Diff. Geometry, 25(1987), 167-197; *CR normal coordinates and the Yamabe problem*, ibidem, 29(1989), 303-344.

[29] Cf. S. Dragomir & D. Perrone, *Levi Harmonic Maps of Contact Riemannian Manifolds*, Journal of Geometric Analysis, (3)24(2014)

[30] Cf. F. Wu & W. Wang, *The Bochner-type formula and the first eigenvalue of the sub-Laplacian on a contact Riemannian manifold*, Differential Geometry and its Applications, 37(2014), 66-88.

[31] Cf. *op. cit.*

[32] Cf. *op. cit.*

[33] Cf. A. Menikoff & J. Sjöstrand, *On the eigenvalues of a class of hypoelliptic operators*, Math. Ann., 235(1978), 55-58.

[34] Cf. W. Wang & F. Wu, *On the Yamabe problem on contact Riemannian manifolds*, Annals of Global Analysis and Geometry, 56(2019), 465-506.

[35] Cf. *op. cit.*

[36] And not enough room is devoted to the mathematical analysis part of the same problem. This situation is aggravated by the already incomplete treatment by D. Jerison & J.M. Lee, *The Yamabe problem on CR manifolds*, J. Diff. Geometry, 25(1987), 167-197; *Intrinsic CR normal coordinates and the CR Yamabe problem* J. Diff. Geometry, 29(1989), 303-343. Of course the gaps in D. Jerison & J.M. Lee's work should be filled in, prior to any attempt to extend their arguments to the non-integrable almost CR structures appearing on contact Riemannian manifolds.

Kohn-Hodge-de Rham theory for the tangential Cauchy-Riemann twisted[37] complex

$$C^\infty(M) \otimes \mathbb{C} \xrightarrow{\overline{\partial}_M} \Omega^{0,1}(M) \xrightarrow{\overline{\partial}_M} \cdots \xrightarrow{\overline{\partial}_M} \Omega^{0,n}(M) \to 0$$

[$\overline{\partial}_M^2 \neq 0$ in general] and its twisted cohomology

$$H^{0,q}_{\overline{\partial}_M}(M) = \frac{\mathrm{Ker}\{\overline{\partial}_M \, : \, \Omega^{0,q}(M) \to \bullet\}}{\left[\overline{\partial}_M \, \Omega^{0,\,q-1}(M)\right] \cap \mathrm{Ker}\{\overline{\partial}_M \, : \, \Omega^{0,q}(M) \to \bullet\}} \, .$$

The problem posed will require recovering J.J. Kohn's subelliptic estimates[38] on a contact Riemannian manifold.

2. E.E. Levi convexity

2.1 Levi form. Let $(M^{2n+k}\, ,\, \mathcal{H})$ be a CR manifold, of type $(n,\, k)$. Let

$$\pi_x : \mathcal{H}_x \to \frac{T_x(M) \otimes_{\mathbb{R}} \mathbb{C}}{\mathcal{H}_x \oplus \overline{\mathcal{H}}_x}\, , \quad x \in M,$$

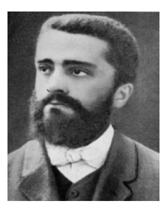
be the projection. The *Levi form* is

$$\mathcal{L}_x : \mathcal{H}_x \to \frac{T_x(M) \otimes_{\mathbb{R}} \mathbb{C}}{\mathcal{H}_x \oplus \overline{\mathcal{H}}_x}\, , \quad \mathcal{L}_x(w) = -\frac{i}{2}\, \pi_x \left[\overline{L},\, L\right]_x ,$$

$$x \in M, \quad w \in \mathcal{H}_x\, , \quad L \in C^\infty(\mathcal{H}), \quad L_x = w.$$

The Levi form is due to E.E. Levi, though confined to (smooth) boundaries of domains $\Omega \subset \mathbb{C}^2$.

Eugenio Elia Levi (1883-1917)



E.E. Levi was born on the 18th of October 1893 in Turin and died in war, shot in the head, at a location near Cormons (Gorizia) on the 28th of October 1917. His death was surely the greatest loss suffered[39] by the Italian mathematics - and not only - due to the 1914-1918 war. **L.** completed his university studies at *Scuola Normale Superiore* of Pisa in 1904 and served there as an assistant of Ulisse Dini. In 1909 **L.** became a professor of infinitesimal analysis at the University of Genova where he remained until he was called for the military service and the successive all too early ending. As F. Tricomi wrote, despite of his premature death (when only 34) **L.** *may be considered* (on the basis of the about thirty works he wrote) *one of the major Italian mathematicians of the twentieth century*. Remarkable are **L.**'s works on second order elliptic partial differential equations (1907-1908) and also his works on the heat equation and on arguments of variational calculus. **L.** also has contributions in differential geometry and group theory. **L.** was a correspondent member of *Accademia Nazionale dei Lincei* (nominated in 1911).

[37] In the sense of I. Vaisman, *New examples of twisted cohomologies*, Boll. Un. Mat. Ital., (7)7(1993), no. 2, 355-368.
[38] Cf. J.J. Kohn, *op. cit.*
[39] The biographical notes on **L.** are based on material by G. Loria and G. Fubini [Boll. Bibl. Storia Mat., (2)1(1918), 38-45] and by C.S. Roero, `http://www.torinoscienza.it/accademia/personaggi`.

In more generality, if $(M^{2n+1}\,,\,\mathcal{H})$ is a CR manifold of type $(n,\,1)$, then let

$$H(M) = \mathrm{Re}\big\{\mathcal{H}\oplus\overline{\mathcal{H}}\big\}$$

be the *Levi distribution*, carrying the complex structure

$$J: H(M)\to H(M),\quad J(Z+\overline{Z}) = i\,(Z-\overline{Z}),\quad Z\in\mathcal{H},$$

and let $H(M)^\perp\subset T^*(M)$ be the *conormal bundle* i.e. the real line bundle

$$H(M)^\perp_x = \big\{\omega\in T^*_x(M)\,:\,\mathrm{Ker}(\omega)\supset H(M)_x\big\},\quad x\in M.$$

If M is connected and orientable[40] the conormal bundle is trivial i.e.

$$H(M)^\perp \approx M\times\mathbb{R}$$

(a vector bundle isomorphism) so that $H(M)^\perp$ admits globally defined, nowhere zero, C^∞ cross sections $\theta\in C^\infty\big(H(M)^\perp\big)$, each of which is commonly referred to as a *pseudohermitian structure* on M. A pseudohermitian structure is then a differential 1-form $\theta\in\Omega^1(M)$ such that $\mathrm{Ker}(\theta) = H(M)$. Let $\mathcal{P}(M,\mathcal{H})$ be the space of all pseudohermitian structures on M. For every $\theta\in\mathcal{P}(M,\mathcal{H})$ we consider the field of bilinear forms

$$G_\theta(X,Y) = (d\theta)(X,\,JY),\quad X,Y\in H(M).$$

As a consequence of formal integrability $G_\theta(JX,\,JY) = G_\theta(X,Y)$. The CR structure $\mathcal{H}$ is *nondegenerate* if G_θ is nondegenerate for some $\theta\in\mathcal{P}(M,\,\mathcal{H})$ (and thus for all). Also $\mathcal{H}$ is *strictly pseudoconvex* if G_θ is positive definite, for some $\theta\in\mathcal{P}(M,\mathcal{H})$. When $\mathcal{H}$ is strictly pseudoconvex, one denotes by $\mathcal{P}_+(M,\mathcal{H})$ the set of all *positively oriented* contact forms θ i.e. such that G_θ is positive definite. Next

$$\Phi_x : \frac{T_x(M)\otimes_{\mathbb{R}}\mathbb{C}}{\mathcal{H}_x\oplus\overline{\mathcal{H}}_x}\to H(M)^\perp_x\,,$$

$$\Phi_x\big(\pi_x(w)\big) := \theta_x(w)\,\theta_x\,,\quad w\in T_x(M)\otimes_{\mathbb{R}}\mathbb{C},\quad x\in M,$$

is a vector bundle isomorphism

$$\Phi : \frac{T(M)\otimes\mathbb{C}}{\mathcal{H}\oplus\overline{\mathcal{H}}}\approx H(M)^\perp$$

and[41]

$$\Phi_x\,\mathcal{L}_x(w) = G_{\theta,\,x}(w,\,\overline{w})$$

for any $w\in\mathcal{H}_x$. One then legitimately refers to G_θ, or its $\mathbb{C}$-linear extension to $H(M)\otimes\mathbb{C}$, as the *Levi form*.

If $\Omega = \{\rho>0\}\subset\mathbb{C}^2$ is a domain with smooth boundary $M = \partial\Omega = \{\rho = 0\}$, the *Levi invariant* of M is

$$\mathcal{L}(\rho) := 2\,G_\theta(L,\,\overline{L}) = \sum_{j,\,k=1}^{2}\frac{\partial^2\rho}{\partial z_j\,\partial\overline{z}_k}\,\lambda_j\,\overline{\lambda}_k\,,\quad \lambda_1 = \frac{\partial\rho}{\partial z_2}\,,\quad \lambda_2 = -\frac{\partial\rho}{\partial z_1}\,,$$

[40] A fact that we assume tacitly to begin with.

[41] Indeed

$$\Phi_x\,\mathcal{L}_x(w) = -\frac{i}{2}\,\Phi_x\,\pi_x\,\big[\overline{L}\,,\,L\big]_x = -\frac{i}{2}\,\theta_x\big(\big[\overline{L}\,,\,L\big]_x\big)\,\theta_x =$$

$$= i\,(d\theta)\big(\overline{L},\,L\big)_x = G_\theta\big(L,\,\overline{L}\big)_x = G_{\theta,\,x}(w,\,\overline{w}).$$

$$\theta = \mathbf{j}^* \Big\{ \frac{i}{2} \big(\overline{\partial} - \partial \big) \rho \Big\}, \;\; \mathbf{j}_* L = - \frac{\partial \rho}{\partial z_2} \frac{\partial}{\partial z_1} + \frac{\partial \rho}{\partial z_1} \frac{\partial}{\partial z_2}\,, \;\; \mathbf{j} : M \hookrightarrow \mathbb{C}^2\,.$$

This is precisely how E.E. Levi introduced[42] the Levi form. We end this section by presenting a simple result due to L. Amoroso[43] demonstrating the Levi form at work i.e. illustrating the influence of the properties of the Levi form of the boundary of a domain $\Omega \subset \mathbb{C}^2$ on the analysis of pluriharmonic functions in Ω.

A C^2 function $u : \Omega \to \mathbb{R}$ is *pluriharmonic* if $\partial \overline{\partial} u = 0$ in Ω.

The real part $u = \frac{1}{2}\big(f + \overline{f}\big)$ of a function f, which is holomorphic in Ω, is pluriharmonic. If Ω is simply connected the converse is true i.e. any pluriharmonic function is the real part of some function holomorphic in Ω. Here we deal with the classical problem to find necessary and sufficient conditions on a function u defined on the boundary of Ω such that u is the boundary values of a pluriharmonic function.

A linear partial differential operator P, of order m, whose coefficients are continuous on $\overline{\Omega}$, is said to be *tangential* to $\partial\Omega$ if $Pu = 0$ on $\partial\Omega$ for any $u \in C^m(\mathbb{C}^2)$ which satisfies $u|_{\partial\Omega} = 0$.

L. Amoroso described[44] boundary values of pluriharmonic functions in terms of the Levi invariant.

Theorem 1. Assume that $\Omega \subset \mathbb{C}^2$ admits a defining function $\rho \in C^2(\Omega)$ such that $\mathcal{L}(\rho) \neq 0$ everywhere on $\partial\Omega$. Then there is a tangential (relative to $\partial\Omega$) second order linear differential operator D such that

$$\frac{\partial u}{\partial \nu} = \frac{1}{\mathcal{L}(\rho)} D u, \tag{9}$$

for any $u \in C^2(\overline{\Omega})$ which is pluriharmonic in Ω.

Of course the Levi invariant makes sense for every 3-dimensional CR manifold $(M, \mathcal{H})$, equipped with a pseudohermitian structure θ, and the prescription "Levi invariant $\neq 0$" is that $\mathcal{H}$ be nondegenerate.

2.2. Levi flat CR manifolds. A CR manifold $(M^{2n+k}, \mathcal{H})$, of type (n, k), is *Levi flat* if $\mathcal{L} = 0$. By a result of T. Levi-Civita[45]

Theorem 2. Every Levi flat CR manifold $(M^{2n+k}, \mathcal{H})$ of type (n, k) is foliated by complex n-dimensional manifolds.

The nowadays proof of Theorem is to observe that $(M, \mathcal{H})$ is Levi flat if and only if $H(M)$ is a completely integrable Pfaffian system, and hence a Levi flat manifold carries a foliation $\mathcal{F}$ by real $2n$ dimensional submanifolds. On the other hand, the complex structure $J : H(M) \to H(M)$ determines an almost complex structure on every leaf $S \in M/\mathcal{F}$, and one may apply the Newlander-Nirenberg theorem[46] to that leaf.

[42] Cf. E.E. Levi, *Studi sui punti singolari essenziali delle funzioni analitiche di due o più variabili complesse*, Annali di Matematica Pura e Applicata, s. III, XVII, n. 1, 1910, pp. 61-87.

[43] Cf. L. Amoroso, *Sopra un problema al contorno*, Rend. Circ. Matem. Palermo, 33(1912), 75-85.

[44] Cf. *op. cit.*

[45] Cf. T. Levi-Civita, *Sulle funzioni di due o più variabili complesse*, Rend. Acad. Naz. Lincei, **14** (1905), 492-499.

[46] The Newlander-Nirenberg theorem (on the integrability of almost complex structures) wasn't known to T. Levi-Civita, at the time Theorem was published (T. Levi-Civita gave a direct proof to the result).

Let $(M,\, \mathcal{H})$ be a Levi-flat CR manifold of type $(n, 1)$, and let $\mathcal{F}$ be the codimension one foliation of M tangent to $H(M)$. Then every local defining submersion of $\mathcal{F}$ is a real valued CR function on M. This exhibits a limit to the analogy between holomorphic functions and CR functions, as any real valued holomorphic function is a constant while the same is true for real valued CR functions only on a nondegenerate CR manifold.

2.3 Nondegenerate CR manifolds. Let $(M,\, \mathcal{H})$ be a CR manifold, of type $(n, 1)$. If $\mathcal{H}$ is nondegenerate then every pseudohermitian structure $\theta \in \mathcal{P}(M,\, \mathcal{H})$ is a contact structure on M i.e. $\theta \wedge (d\theta)^n$ is a volume form. As shown by S.M. Webster[47] every nondegenerate CR manifold $(M, \mathcal{H})$, on which a contact form θ has been fixed, admits a rich geometric structure e.g. a semi-Riemannian metric g_θ given by

$$\begin{gathered} g_\theta = G_\theta \ \text{ on } \ H(M) \otimes H(M), \\ g_\theta(X, T) = 0, \quad g_\theta(T, T) = 1, \quad X \in H(M), \\ T \in \mathfrak{X}(M), \quad \theta(T) = 1, \quad T \rfloor d\theta = 0, \end{gathered}$$

[g_θ is the *Webster metric*, and T is the *Reeb vector* field of (M, θ)] which is Riemannian when $\mathcal{H}$ is strictly pseudoconvex and θ is positively oriented. An adapted linear connection ∇ (the *Tanaka-Webster connection*, similar to both the Levi-Civita connection of a Riemannian manifold, and the Chern connection of a Hermitian manifold), parallelizing the Levi distribution $H(M)$, its complex structure J, and the Webster metric g_θ, may then be built

$$\begin{gathered} X \in \mathfrak{X}(M), \quad Y \in C^\infty(H(M)) \Longrightarrow \nabla_X Y \in C^\infty(H(M)), \\ \nabla J = 0, \quad \nabla g_\theta = 0, \\ \mathrm{Tor}_\nabla(Z,\, W) = 0, \quad \mathrm{Tor}_\nabla(Z,\, \overline{W}) = 2i\, G_\theta(Z,\, \overline{W}), \\ \tau \circ J + J \circ \tau = 0, \quad \tau(X) := \mathrm{Tor}_\nabla(T,\, X), \\ Z,\, W \in \mathcal{H}, \quad X \in \mathfrak{X}(M). \end{gathered}$$

Also, if $\mathcal{H}$ is strictly pseudoconvex then the total space of the canonical circle bundle $S^1 \to C(M) \xrightarrow{\pi} M$ carries a natural Lorentzian metric $F_\theta \in \mathrm{Lor}\big(C(M)\big)$ (the *Fefferman metric*) such that the projection $\pi : (C(M),\, F_\theta) \to (M,\, g_\theta)$ enjoys properties similar to those of a semi-Riemannian submersion, though having degenerate fibers.

3. CR embeddability problem

Let $(M,\, \mathcal{H})$ be a CR manifold, of type $(n,\, 1)$.

Definition 4. The *local CR embeddability* problem is whether, given a point $x \in M$, there is an open neighborhood $x \in U \subset M$ and a C^∞ immersion $\phi : U \to \mathbb{C}^{n+1}$ such that $\phi : U \to N := \phi(U)$ is a CR isomorphism of the CR manifolds $\big(U,\, \mathcal{H}_U\big)$ and $\big(N,\, T_{1,0}(N)\big)$. The *global* CR embeddability problem is whether a globally defined C^∞ immersion $\phi : M \to \mathbb{C}^{n+1}$ of the sort[48] exists. Let $x \in M$ and let us assume that the local CR embeddability problem has solution i.e. there is a C^∞ immersion $\phi : U \to \mathbb{C}^{n+1}$, as in the previous definition. Let $\{L_\alpha \,:\, 1 \leq \alpha \leq n\}$ be a local frame of $\mathcal{H}$, about[49] the point x. Then

$$\phi_* \, L_\alpha \in \phi^{-1}\, T_{1,0}(N) \subset \phi^{-1} T'(\mathbb{C}^{n+1}),$$

[47] Cf. *op. cit.*
[48] That is, such that $\phi : M \to N := \phi(M)$ is a CR isomorphism.
[49] Perhaps defined on some smaller open neighborhood of x in U, that we denote again by U and restrict ϕ to it.

$$\phi_* L_\alpha = L_\alpha\big(\phi^j\big)\left(\frac{\partial}{\partial z^j}\right)^\phi + L_\alpha\big(\overline{\phi^j}\big)\left(\frac{\partial}{\partial \overline{z}^j}\right)^\phi,$$

hence

$$\overline{L}_\alpha\left(\phi^j\right) = 0 \tag{10}$$

i.e. ϕ^j are CR functions on U. So the local CR embeddability problem is to produce enough functionally independent solutions ϕ^j to the tangential C-R equations $\overline{L}_\alpha u = 0$ [so that to build a CR immersion $\phi = \left(\phi^1, \cdots, \phi^{n+1}\right) : U \to \mathbb{C}^{n+1}$]. However, the innocent looking like (local) CR embeddability problem[50] turns out to be quite intricate, and cannot be solved in general, that is examples of CR manifolds which aren't locally embeddable do exist, as we shall shortly see.

By a (classical) result of A. Andreotti & C.D. Hill[51] any real analytic CR manifold is locally embeddable. In the C^∞ category the CR embedding problem was largely solved by M. Kuranishi[52], who showed that every strictly pseudoconvex CR manifold of real dimension $2n+1 \geq 9$ is locally embeddable in $\mathbb{C}^{n+1}$. T. Akahori settled[53] the question in dimension 7.

3.1. Nirenberg's non embeddable example. Let $\mathbb{H}_1 = \mathbb{C} \times \mathbb{R}$ be the 3-dimensional Heisenberg group, with the CR structure spanned by $L \equiv \partial/\partial z + i\,\overline{z}\,\partial/\partial t$. L. Nirenberg built[54] a function $\varphi \in C^\infty\big(\mathbb{H}_1\big)$ such that the CR structure spanned by $L + \varphi\,\partial/\partial t$ isn't embeddable in any neighborhood of the origin in $\mathbb{H}_1$. The proof is to show that no pair of CR functions $\{u^1, u^2\}$ defined in a neighborhood $U \subset \mathbb{H}_1$ are functionally independent, so that there is no CR immersion $\left(u^1, u^2\right) : U \to \mathbb{C}^2$. Nirenberg's argument is to consider a sequence of open solid tori $T_{\delta_\nu} \subset \mathbb{H}_1$ tending to the origin $T_{\delta_\nu} \to 0$ as $\nu \to \infty$, together with a choice of φ such that $\varphi > 0$ on each T_{δ_ν} and $\varphi = 0$ outside $\bigcup_{\nu=1}^\infty T_{\delta_\nu}$. Then

$$\int_{\phi(\partial T_{\delta_\nu})} \left(u \circ \phi^{-1}\right) dw \wedge dz = 0, \quad \nu \geq 1,$$

for any CR function $u : \mathbb{H}_1 \to \mathbb{C}$, of class C^1, with respect to the new CR structure $\mathcal{H}(\varphi)$, thus yielding (by Stokes' theorem)

$$\int_{T_{\delta_\nu}} \varphi\,\frac{\partial u}{\partial t}\, dt \wedge dz \wedge d\overline{z} = 0.$$

This shows that $(\partial u/\partial t)_0 = 0$ and then $(\partial u/\partial \overline{z})_0 = 0$, because u is a CR function with respect to the canonical CR structure $\mathcal{H}(0)$ wherever $\varphi = 0$. Consequently $(du^1 \wedge du^2)_0 = 0$ for any pair of CR functions u^j on $(\mathbb{H}_1, \mathcal{H}(\varphi))$ so that $(u^1, u^2) : U \to \mathbb{C}^2$ fails to be an immersion (for an arbitrary neighborhood $U \subset \mathbb{H}_1$ of $0 \in \mathbb{H}_1$).

L. Nirenberg's non embeddable example was generalized to the vector valued case (i.e. for CR functions $u : U \to \mathfrak{X}$, with values in a complex Fréchet space, of Theodoresco class B^1) by E. Barletta et al.[55].

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[50] Introduced in mathematical practice by J.J. Kohn, *Boundaries of complex manifolds*, Proc. Conf. on Complex Analysis, Minneapolis, 1964, springer-Verlag, New York, 1965, pp. 81-94.

[51] Cf. A. Andreotti & C.D. Hill, *Complex characteristic coordinates and the tangential Cauchy-Riemann equations*, Ann. Scuola Norm. Sup., Pisa, 26(1972), 299-324.

[52] Cf. M. Kuranishi, *Strongly pseudoconvex CR structures over small balls*, I-III, Ann. of Math., 115(1982), 451-500; ibidem, 116(1982), 1-64; ibidem, 116(1982), 249-330.

[53] Cf. T. Akahori, *A new approach to the local embedding theorem of CR structures, the local embedding theorem for $n \geq 4$*, Amer. Math. Soc. Memoires, No. 366, 1987.

[54] Cf. L. Nirenberg, *On a question of Hans Lewy*, Russian Mathematical Surveys, 29(1974), 251-262.

[55] E. Barletta & S. Dragomir & F. Esposito & I.D. Platis, *On Nirenberg's non-embeddable CR structure*, Complex Variables and Elliptic Equations, 2021, `https://doi.org/10.1080/17476933.2021.198603`.

It is unknown whether Nirenberg's construction admits a several complex variables version (on the Heisenberg group $\mathbb{H}_n$ with $n \geq 2$).

Another open problem is to transplant Nirenberg's construction to compact quotients $\mathbb{H}_1/G(s)$, $0 < s < 1$, by properly discontinuous actions [$G(s) = \{\delta_s^m \, : \, m \in \mathbb{Z}\}$ is the discrete group generated by the parabolic dilation $\delta_s(z,t) = (sz, \, s^2 t)$, $(z,t) \in \mathbb{H}_1$] as considered by S. Dragomir[56] for different purposes[57].

3.2. Hill's example *versus* the unsolvability of the Lewy operator. Let $\overline{L} \equiv \partial/\partial\overline{z} - i\, z\, \partial t$ be the Lewy operator and let us consider the first order PDE

$$\overline{L}\chi = \omega \tag{11}$$

where $\omega : \mathbb{H}_1 \to \mathbb{C}$ is a C^∞ function.

Definition 5. We say (11) is *solvable* at a point $(z_0\,,\, t_0) \in \mathbb{H}_1$ if there is an open set $U \subset \mathbb{H}_1$ such that $(z_0\,,\, t_0) \in U$ and there is a C^∞ function $\chi : U \to \mathbb{C}$ such that $\overline{L}\chi = \omega$ on U.

Let $M = \mathbb{H}_1 \times \mathbb{C}$ endowed with the Cartesian coordinates $(z, \, t, \, \zeta)$, and let us consider the complex vector fields

$$P, \; Q \in C^\infty\big(T(M) \otimes \mathbb{C}\big), \quad P \equiv \frac{\partial}{\partial \overline{\zeta}}\,, \quad Q \equiv \overline{L} + \omega(z,t)\, \frac{\partial}{\partial \zeta}\,.$$

Then $\{P, \, Q\}$ span a CR structure $\mathcal{H}_\omega$ on M, of type $(2, \, 1)$. By a result of C.D. Hill[58] the CR structure $\mathcal{H}_\omega$ is locally embeddable at $(z_0\,,\, t_0\,,\, \zeta_0) \in M$ if and only if (11) is solvable at $(z_0\,,\, t_0)$ [hence the unsolvability of the Lewy operator (the unsolvability of (11) for non real analytic choices of ω) produces examples of 5-dimensional CR manifolds M which aren't locally embeddable].

We ought to recall that Lewy's proof[59] of the unsolvability of $\overline{L}$ is merely that the solvability of $\overline{L}\chi = \omega$ at a point requires that ω be real analytic at that point. There is however much more[60] in the work by H. Lewy[61] i.e. one builds a family $\{\omega_\epsilon\}_{\epsilon \in B \setminus E}$ of "free terms" (in fact, Baire category[62] many) such that the equation $\overline{L}\chi = \omega_\epsilon$ has no solution.

Precisely, let $\psi \in C^\infty(\mathbb{R})$ be a real valued, periodic, function that isn't real analytic at any $t \in \mathbb{R}$. Also, let $Q_\nu = (\xi_\nu\,,\, \eta_\nu\,,\, \zeta_\nu) \in \mathbb{R}^3$ be a sequence of points, which is dense in $\mathbb{R}^3$, and let us set

$$c_\nu = 2^{-\nu}\, \exp\big(-\rho_\nu\big), \qquad \rho_\nu = |\xi_\nu| + |\eta_\nu|.$$

Also, let B be the Banach space of all bounded infinite sequences $\epsilon = \{\epsilon_\nu\}_{\nu \geq 1}$ of real numbers $\epsilon_\nu \in \mathbb{R}$, with the norm $\|\epsilon\| = \sup_{\nu \geq 1} \big|\epsilon_\nu\big|$. The announced functions ω_ϵ are the sums of the series

$$\omega_\epsilon(x, \, y, \, t) = \sum_{\nu=1}^\infty \epsilon_\nu\, c_\nu\, \psi'\big(t - 2\,\eta_\nu\, x + 2\,\xi_j\, y\big)\,, \quad \epsilon \in B.$$

[56]S. Dragomir, *On a conjecture of J.M. Lee*, Hokkaido Math. J., (1)23(1994), 35-49.

[57]To provide examples and counterexamples *vis-a-vis* to the *Lee conjecture*.

[58]Cf. C.D. Hill, *What is the notion of a complex manifold with smooth boundary?*, Algebraic Analysis, 1(1988), 185-201.

[59]Cf. H. Lewy, *op. cit.*

[60]In contrast with the opinion expressed at the time by Gaetano Fichera [Italian mathematician (1922-1996)] during one of his many personal crusades.

[61]Cf. F. John, *Partial Differential Equations*, Applied Mathematical Sciences, Vol. 1, Springer-Verlag, New York-Heidelberg-Berlin, 1982 (fourth edition), pp. 235-239.

[62]A subset $E \subset S$ of a topological space is *nowhere dense* if its closure $\overline{E}$ has an empty interior. A subset $E \subset S$ is a set of the *first category* if E is a countable union of nowhere dense sets. A subset $E \subset S$ is a set of the *second category* if it isn't a set of the first category. *Baire's Theorem* is that for every complete metric space S the intersection of every countable family of dense open subsets of S is dense in S. As a corollary of Baire's Theorem, every complete metric space is of the second category in itself.

Let us explain the rather vague "Baire category" statement, used by us above. To this end, let $\Omega_{\nu,n} = B_{1/\sqrt{n}}\big(Q_\nu\big) \subset \mathbb{R}^3$ be the ball of center $Q_\nu = (\xi_\nu\, , \eta_\nu\, , \zeta_\nu)$ and radius $1/\sqrt{n}$. Next, let $E_{\nu,n} \subset B$ consist of all $\epsilon \in B$ for which there is a solution $\chi \in C^1\big(\Omega_{\nu,n}\big)$ to the equation

$$\overline{L}\chi = \omega_\epsilon \tag{12}$$

such that[63]

$$\chi(Q_\nu) = 0, \tag{13}$$

$$\big|D^\alpha \chi(P)\big| \le n, \quad |\alpha| \le 1, \quad P \in \Omega_{\nu,n}\,, \tag{14}$$

$$\big|D^\alpha \chi(P) - D^\alpha \chi(Q)\big| \le n|P-Q|^{1/n}\,, \tag{15}$$
$$|\alpha| = 1, \;\; P,\, Q \in \Omega_{\nu,n}\,.$$

By a result of H. Lewy[64] the sets $E_{\nu,n}$ are closed subsets of B that are nowhere dense[65]. By Baire's Theorem B is a set of the second category in itself, hence the inclusion

$$E := \bigcup_{\substack{\nu \ge 1 \\ n \ge 1}} E_{\nu,n} \subset B$$

is strict. By Hill's result[66] for every $\epsilon \in B \setminus E$ the CR structure $\mathcal{H}_{\omega_\epsilon}$ isn't locally embeddable at any of the points Q_ν, $\nu \ge 1$.

No extension of C.D. Hill's result[67] to the vector valued case is known, so far. C.D. Hill's example ought to be revisited (in the vector valued setting), as prompted by the discussion of Lewy's unsolvability phenomenon due to E. Barletta & S. Dragomir[68].

3.3. Rossi's spheres. Let $S^3 = \big\{(z,w) \in \mathbb{C}^2 \,:\, z\overline{z} + w\overline{w} = 1\big\}$. The canonical CR structure

$$T_{1,0}(S^3) = \big[T(S^3) \otimes \mathbb{C}\big] \cap T'(\mathbb{C}^2)$$

is the span of $L_0 \equiv \overline{w}\,\partial/\partial z - \overline{z}\,\partial/\partial w$. Let us consider the first order differential operators

$$L_t = L_0 + t\,\overline{L}_0\,, \quad |t| < 1,$$

and let $\mathcal{H}(t)$ be the CR structure on S^3 spanned by L_t, so that $\big(S^3\,,\,\mathcal{H}(t)\big)$ with $|t| < 1$, $t \neq 0$, are the *Rossi spheres*[69], known to be non globally embeddable i.e. there is no CR isomorphism $\phi : (S^3,\, \mathcal{H}(t)) \to N$ of a Rossi sphere onto a real hypersurface $N \subset \mathbb{C}^2$.

Given a strictly pseudoconvex CR manifold M, endowed with a positively oriented contact form, E. Barletta et al.[70] introduced a natural weakening of the global CR embeddability problem, seeking for a at least K-quasiconformal map $\phi : M \to N \subset \mathbb{C}^2$ i.e. a contact transformation

$$(d_x\phi)H(S^3)_x = H(N)_{\phi(x)}\,, \quad x \in S^3\,,$$

[63] Condition (14) represents bounds for χ and its first order derivatives in $\Omega_{\nu,n}$, while (15) prescribes a uniform Hölder condition on the first order derivatives. It should be mentioned the Lewy's unsolvability result in its original formulation stated the existence of $\omega \in C^\infty(\mathbb{R}^3)$ such that the equation $\overline{L}\chi = \omega$ has no solution whose domain is an open set $\Omega \subset \mathbb{R}^3$, with $\chi \in C^1(\Omega)$ **and** χ_x, χ_y, χ_z **Hölder continuous** in Ω. The "Hölder continuity" requirement was later on removed by P. Hartman, *On smooth linear partial differential equations without solutions*, Proc. Amer.Math. Soc., 10(1959), pp. 252-257.

[64] Cf. *op. cit.*

[65] That is $E_{\nu,n}$ has no interior points (cf. also our previous footnote).

[66] Cf. *op. cit.*

[67] Cf. *op. cit.*

[68] Cf. E. Barletta & S. Dragomir, *On Lewy's unsolvability phenomenon*, Complex Var. Elliptic Equations, (9)57(2012), 971-981.

[69] Cf. H. Rossi, *Attaching analytic spaces to an analytic space along a pseudoconcave boundary*, In Proceedings of the Conference on Complex Analysis, Minneapolis, MA, USA, 16-21 March 1964; Springer: Berlin, Germany, 1965; pp. 242–256.

[70] Cf. E. Barletta & S. Dragomir & F. Esposito, *Beltrami Equations on Rossi Spheres*, Mathematics, 2022, 10, 371. `https://doi.org/10.3390/math10030371`.

such that for any $X \in H(S^3)$

$$\frac{1}{K}\, G_{\theta,\,\mathcal{H}(t)}(X, X) \le \frac{G^{\phi}_{\Theta}(\phi_* X,\, \phi_* X)}{\lambda(f;\, \theta,\, \Theta)} \le K\, G_{\theta,\,\mathcal{H}(t)}(X, X),$$

$$\theta = \frac{i}{2}\big(\overline{\partial} - \partial\big)(|z|^2 + |w|^2) \in \mathcal{P}_+(S^3,\, T_{1,0}(S^3)) = \mathcal{P}_+(S^3\,,\, \mathcal{H}(t)),$$
$$\Theta \in \mathcal{P}_+(N,\, T_{1,0}(N)), \quad \phi^*\Theta = \lambda(f;\, \theta, \Theta)\, \theta,$$

the finding of which amounts to solving the Beltrami equations, as derived by A. Korányi & H.M. Reimann[71]. When $M \in \big\{(S^3,\, \mathcal{H}(t)) \ : \ |t| < 1, \quad t \neq 0\big\}$ E. Barletta et al.[72] solved the Beltrami equations

$$\overline{L}_t(g) = \mu(\,\cdot\,, t)\, L_t(g)$$

on S^3 [by using the Greiner-Kohn-Stein solution to the Lewy equation, and the Bargmann representations of the Heisenberg group] for Sobolev-type solutions g_t such that $g_t - v \in W^{1,2}_F(S^3\,,\, \theta)$ with $v \in \mathrm{CR}^\infty(S^3\,,\, \mathcal{H}(0))$. The weakened CR embeddability problem, as proposed by E. Barletta et al.[73], is otherwise open.

[71]Cf. A. Korányi & H.M. Reimann, *Quasiconformal mappings on CR manifolds*, In Conference in Honor of Edoardo Vesentini; Springer: Berlin, Germany, 1988; pp. 59-75.

[72]Cf. *op. cit.*

[73]Cf. *op. cit.*

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On Some Generalized Einstein Metric Conditions

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Abstract

Let (M, g) be a semi-Riemannian manifold. We denote by g, R, S, κ and C the metric tensor, the Riemann-Christoffel curvature tensor, the Ricci tensor, the scalar curvature and the Weyl conformal curvature tensor of (M, g), respectively. Using these tensors we can define the $(0, 6)$-tensors $R \cdot R$, $R \cdot C$, $C \cdot R$, $C \cdot C$ and $Q(A, T)$, where A is a symmetric $(0, 2)$-tensor and T a generalized curvature tensor (see, e.g., [32], [35] and [48]).

A semi-Riemannian manifold (M, g), $\dim M = n \geq 2$, is said to be an *Einstein manifold* [4], or an *Einstein space*, if at every point of M its Ricci tensor S is proportional to g, i.e., $S = (\kappa/n)\, g$ on M, assuming that κ is constant when $n = 2$. According to [4, p. 432], this condition is called the *Einstein metric condition*. Einstein manifolds form a natural subclass of several classes of semi-Riemannian manifolds which are determined by curvature conditions imposed on their Ricci tensor [4, Table, pp. 432-433]. These conditions are called *generalized Einstein metric conditions* [4, Chapter XVI].

The tensor $R \cdot C - C \cdot R$ of every semi-Riemannian Einstein manifold (M, g), $n \geq 4$, satisfies the following identities [44, Theorem 3.1] (see also [29, p. 100001-1] and [48, p. 107])

$$\begin{aligned} R \cdot C - C \cdot R = \frac{\kappa}{(n-1)n}\, Q(g, R) &= \frac{\kappa}{(n-1)n}\, Q(g, C) \\ &= \frac{1}{n-1}\, Q(S, R) \\ &= \frac{1}{n-1}\, Q(S, C), \end{aligned}$$

$$R \cdot C - C \cdot R = \frac{\kappa}{n-1}\, Q(g, C) - Q(S, C) = Q\left(\frac{\kappa}{n-1}\, g - S, C\right). \qquad (1)$$

We can express the tensor $R \cdot C - C \cdot R$ of some non-Einstein and non-conformally flat semi-Riemannian manifolds (M, g), $\dim M \geq 4$, as a linear combination of

$(0,6)$-Tachibana tensors $Q(A,T)$, e.g., $A = g$ or $A = S$ and $T = R$ or $T = C$. These conditions form a family of generalized Einstein metric conditions. Semi-Riemannian manifolds, and in particular hypersurfaces isometrically immersed in spaces of constant curvature, satisfying such conditions were investigated in several papers, see, e.g., [2, 12, 28, 32, 34, 35, 45, 48]. We refer to [26] (see also [29]) for a survey of results on manifolds (hypersurfaces) satisfying such conditions.

If a non-quasi-Einstein and non-conformally flat semi-Riemannian manifold (M,g), $n \geq 4$, satisfies the following two curvature conditions of pseudosymmetry type: $R \cdot R = L_1\, Q(g,R)$ and $C \cdot C = L_2\, Q(g,C)$, where L_1 and L_2 are some functions, then the curvature tensor R is a linear combination of the Kulkarni-Nomizu products formed by the metric tensor g and the Ricci tensor S [53, Theorem 3.1, Teorem 3.2 (ii)]. A non-quasi-Einstein and non-conformally flat semi-Riemannian manifold (M,g), $n \geq 4$, with curvature tensor R expressed by the above-mentioned linear combination of the Kulkarni-Nomizu products is called a *Roter type manifold*, or a *Roter manifold*, or a *Roter space* (see, e.g., [10, Section 15.5], [26, 32, 35], [48, Section 4]). Every Roter space satisfies (1) (see, e.g., [32, Proposition 3.3], [35, Theorem 2.4 (ii)]).

Let M, $\dim M \geq 4$, be a hypersurface isometrically immersed in a space of constant curvature such that at every point M has exactly two distinct principal curatures, λ_1 with multiplicity p and λ_2 with multiplicity $n-p$, $2 \leq p \leq n-2$. If $(p_1 - 1)\lambda_1 + (n-p-1)\lambda_2 \neq 0$, then M is a Roter space, and in a consequence (1) holds on M [27, Theorem 3.3].

Let M, $\dim M \geq 4$, be a hypersurface isometrically immersed in a space of constant curvature such that at every point M has exactly three distinct principal curatures. If the condition $R \cdot C - C \cdot R = Q(g,T)$, where T is a generalized curvature tensor, is satisfied on M, then the tensor T is a linear combination of the curvature tensor R and Kulkarni-Nomizu products formed by the metric tensor g, the Ricci tensor S and its square S^2 (cf. [34, Theorem 5.2]).

If at every point of a non-quasi-Einstein and non-conformally flat hypersurface M, $\dim M \geq 4$, isometrically immersed in a semi-Riemannian space of constant curvature the tensor $R \cdot C - C \cdot R$ is a linear combination of the tensors $Q(g,C)$ and $Q(S,C)$, then (1) holds on M [35, Theorem 5.4].

1. Basic formulas

Let (M,g) be a connected n-dimensional, $n = \dim M \geq 3$, semi-Riemannian manifold of class C^∞ and ∇ its Levi-Civita connection. We define on M the endomorphisms $X \wedge_A Y$, $\mathcal{R}(X,Y)$ and $\mathcal{C}(X,Y)$ by

$$
\begin{aligned}
(X \wedge_A Y)Z &= A(Y,Z)X - A(X,Z)Y,\\
\mathcal{R}(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,\\
\mathcal{C}(X,Y) &= \mathcal{R}(X,Y) - \frac{1}{n-2}\,(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y) - \frac{\kappa}{(n-2)(n-1)}\, X \wedge_g Y,
\end{aligned}
$$

respectively, where $\mathfrak{X}(M)$ is the Lie algebra of vector fields of M, $X, Y, Z \in \mathfrak{X}(M)$, A a symmetric $(0,2)$-tensor, S the Ricci tensor, $\mathcal{S}$ the Ricci operator,

$$\begin{aligned} S(X,Y) &= \operatorname{tr}\{Z \to \mathcal{R}(Z,X)Y\}, \\ g(\mathcal{S}X,Y) &= S(X,Y), \end{aligned}$$

and $\kappa = \operatorname{tr}\mathcal{S}$ the scalar curvature. The Riemann-Christoffel curvature tensor R, the Weyl conformal curvature tensor C and the $(0,4)$-tensor G of (M,g) are defined by

$$\begin{aligned} & R(X_1,X_2,X_3,X_4) = g(\mathcal{R}(X_1,X_2)X_3,X_4), \\ & C(X_1,X_2,X_3,X_4) = g(\mathcal{C}(X_1,X_2)X_3,X_4) \\ = \ & R(X_1,X_2,X_3,X_4) - \frac{1}{n-2}(g(X_1,X_4)\,S(X_2,X_3) - g(X_1,X_3)\,S(X_2,X_4) \\ & +g(X_1,X_4)\,S(X_2,X_3) - g(X_1,X_3)\,S(X_2,X_4)) \\ & +\frac{\kappa}{(n-2)(n-1)}\,(g(X_1,X_4)\,g(X_2,X_3) - g(X_1,X_3)\,g(X_2,X_4)), \\ & G(X_1,X_2,X_3,X_4) = g((X_1 \wedge_g X_2)X_3,X_4) \\ = \ & g(X_1,X_4)\,g(X_2,X_3) - g(X_1,X_3)\,g(X_2,X_4), \end{aligned}$$

respectively, where $X_1,\ldots,X_4 \in \mathfrak{X}(M)$.

For symmetric $(0,2)$-tensors E and F we define their *Kulkarni-Nomizu product* $E \wedge F$ by

$$\begin{aligned} (E \wedge F)(X_1,X_2,X_3,X_4) &= E(X_1,X_4)F(X_2,X_3) + E(X_2,X_3)F(X_1,X_4) \\ &\quad -E(X_1,X_3)F(X_2,X_4) - E(X_2,X_4)F(X_1,X_3), \end{aligned}$$

where $X_1,\ldots,X_4 \in \mathfrak{X}(M)$. Now we can express the Weyl conformal curvature tensor C by

$$C \ = \ R - \frac{1}{n-2}\,g \wedge S + \frac{\kappa}{(n-2)(n-1)}\,G, \quad G \ = \ \frac{1}{2}\,g \wedge g\,.$$

For a symmetric $(0,2)$-tensor E and an $(0,k)$-tensor T, $k \geq 3$, we define their *Kulkarni-Nomizu product* $E \wedge T$ by (see, e.g., [24])

$$\begin{aligned} & (E \wedge T)(X_1,X_2,X_3,X_4,Y_3,\ldots,Y_k) \\ = \ & E(X_1,X_4)T(X_2,X_3,Y_3,\ldots,Y_k) + E(X_2,X_3)T(X_1,X_4,Y_3,\ldots,Y_k) \\ & -E(X_1,X_3)T(X_2,X_4,Y_3,\ldots,Y_k) - E(X_2,X_4)T(X_1,X_3,Y_3,\ldots,Y_k), \end{aligned}$$

where $X_1,\ldots,X_4,Y_3,\ldots,Y_k \in \mathfrak{X}(M)$.

For a symmetric $(0,2)$-tensor A and a $(0,k)$-tensor T, $k \geq 1$, we define the $(0,k+2)$-tensors $R \cdot T$, $C \cdot T$ and $Q(A,T)$ by

$$\begin{aligned} & (R \cdot T)(X_1,\ldots,X_k;X,Y) \ = \ (\mathcal{R}(X,Y) \cdot T)(X_1,\ldots,X_k) \\ = \ & -T(\mathcal{R}(X,Y)X_1,X_2,\ldots,X_k) - \cdots - T(X_1,\ldots,X_{k-1},\mathcal{R}(X,Y)X_k), \\ & (C \cdot T)(X_1,\ldots,X_k;X,Y) \ = \ (\mathcal{C}(X,Y) \cdot T)(X_1,\ldots,X_k) \\ = \ & -T(\mathcal{C}(X,Y)X_1,X_2,\ldots,X_k) - \cdots - T(X_1,\ldots,X_{k-1},\mathcal{C}(X,Y)X_k), \\ & Q(A,T)(X_1,\ldots,X_k;X,Y) \ = \ ((X \wedge_A Y) \cdot T)(X_1,\ldots,X_k) \\ = \ & -T((X \wedge_A Y)X_1,X_2,\ldots,X_k) - \cdots - T(X_1,\ldots,X_{k-1},(X \wedge_A Y)X_k), \end{aligned}$$

respectively. Setting in the above formulas $T = R$, $T = S$, $T = C$, $A = g$ or $A = S$ we obtain the tensors: $R \cdot R$, $R \cdot C$, $C \cdot R$, $C \cdot C$, $R \cdot S$ and $C \cdot S$, and $Q(g,R)$, $Q(g,C)$, $Q(S,R)$, $Q(S,C)$ and $Q(g,S)$.

Let A be a symmetric $(0,2)$-tensor and T a $(0,k)$-tensor. The tensor $Q(A,T)$ is called the *Tachibana tensor of* A *and* T, or, briefly, the *Tachibana tensor* [34]. We like to point out that in some papers the tensor $Q(g,R)$ is called the *Tachibana tensor* (see, e.g., [60, 62, 63]).

Let (M,g), $n \geq 4$, be a semi-Riemannian manifold. We define the following subsets of M: $\mathcal{U}_R = \{x \in M \,|\, R \neq \frac{\kappa}{2(n-1)n}\, g \wedge g \text{ at } x\}$, $\mathcal{U}_S = \{x \in M \,|\, S \neq \frac{\kappa}{n}\, g \text{ at } x\}$ and $\mathcal{U}_C = \{x \in M \,|\, C \neq 0 \text{ at } x\}$. We note that (see, e.g., [25, p. 151])

$$\mathcal{U}_S \cup \mathcal{U}_C = \mathcal{U}_R.$$

2. Pseudosymmetric, Ricci-pseudosymmetric and Weyl-pseudosymmetric manifolds

A semi-Riemannian manifold (M,g), $n \geq 3$, is said to be *pseudosymmetric* if at every point of M the tensors $R \cdot R$ and $Q(g,R)$ are linearly dependent (see, e.g., [1, 11, 19, 36, 39, 71, 82]). The manifold (M,g) is pseudosymmetric if and only if

$$R \cdot R = L_R\, Q(g,R) \tag{2}$$

on $\mathcal{U}_R$, where L_R is some function on this set. Every *semisymmetric* manifold ($R \cdot R = 0$ [79, 80, 81]) is pseudosymmetric. The converse statement is not true.

We mention that pseudosymmetric manifolds admitting concircular vector fields were investigated in [36].

It is well-known that the Schwarzschild spacetime was discovered in 1916 by Schwarzschild and independently by Droste during their study on solutions of Einstein's equations (see, e.g., [74]). That spacetime is a non-semisymmetric, pseudosymmetric spacetime [50]. It seems that the Schwarzschild spacetime, the Kottler spacetime, the Reissner-Nordström spacetime, as well as some FLRW spacetimes (Friedmann-Lemaître-Robertson-Walker spacetimes) are the "oldest" examples of non-semisymmetric pseudosymmetric warped product manifolds (cf. [39, 59]).

It is known that hypersurfaces isometrically immersed in spaces of constant curvature with exactly two distinct principal curvatures at every point are pseudosymmetric [51]. Thus in particular, quasi-umbilical hypersurfaces isometrically immersed in spaces of constant curvature are pseudosymmetric (see, e.g., [39]). We also note that in [57] a special subclass of semisymmetric warped products was investigated. Among other things it was proved (see the proof of Lemma 3 of [57]) that the fiber $(\widetilde{N}, \widetilde{g})$ with $\dim \widetilde{N} \geq 3$, of a semisymmetric warped product manifold $\overline{M} \times_F \widetilde{N}$ satisfies (1).

According to [68] (see also [69, 70]), a pseudosymmetric manifold (M,g), $n \geq 3$, ($R \cdot R = L_R\, Q(g,R)$) is said to be *pseudosymmetric space of constant type* if the function L_R is constant on $\mathcal{U}_R \subset M$.

Theorem 2.1(cf. [15]) Every type number two hypersurface M isometrically immersed in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 3$, is a pseudosymmetric space of constant type. Precisely,

$$R \cdot R = \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g,R)$$

on $\mathcal{U}_R \subset M$, where $\widetilde{\kappa}$ is the scalar curvature of the ambient space.

A semi-Riemannian manifold (M,g), $n \geq 3$, is said to be *Ricci-pseudosymmetric* if at every point of M the tensors $R \cdot S$ and $Q(g,S)$ are linearly dependent (see, e.g., [19]). The manifold (M,g) is

Ricci-pseudosymmetric if and only if

$$R \cdot S = L_S \, Q(g, S) \tag{3}$$

on $\mathcal{U}_S$, where L_S is some function on this set. Every *Ricci-semisymmetric* manifold ($R \cdot S = 0$) is Ricci-pseudosymmetric. The converse statement is not true. According to [54], a Ricci-pseudosymmetric manifold (M, g), $n \geq 3$, ($R \cdot S = L_S \, Q(g, S)$) is said to be *Ricci-pseudosymmetric manifold of constant type* if the function L_S is constant on $\mathcal{U}_S \subset M$.

Theorem 2.2 (cf. [52]) If M is a hypersurface isometrically immersed in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 3$, such that at every point of M there are principal curvatures $0, \ldots, 0, \lambda, \ldots, \lambda, -\lambda, \ldots, -\lambda$, with the same multiplicity of λ and $-\lambda$, and λ is a positive function on M, then M is a Ricci-pseudosymmetric manifold of constant type. Precisely,

$$R \cdot S = \frac{\widetilde{\kappa}}{n(n+1)} \, Q(g, S)$$

on M. In particular, every Cartan hypersurface is a Ricci-pseudosymmetric manifold of constant type.

A semi-Riemannian manifold (M, g), $n \geq 4$, is said to be *Weyl-pseudosymmetric* if at every point of M the tensors $R \cdot C$ and $Q(g, C)$ are linearly dependent (see, e.g., [37]). The manifold (M, g) is Weyl-pseudosymmetric if and only if

$$R \cdot C = L_C \, Q(g, C) \tag{4}$$

on $\mathcal{U}_C$, where L_C is some function on this set. It is easy to check that every pseudosymmetric manifold ($R \cdot R = L_R \, Q(g, R)$) is Weyl-pseudosymmetric ($R \cdot C = L_R \, Q(g, C)$). In particular, every semisymmetric manifold ($R \cdot R = 0$) is Weyl-semisymmetric ($R \cdot C = 0$). If $\dim M \geq 5$ the converse statements are true. Precisely, if $R \cdot C = L_C \, Q(g, C)$, resp., $R \cdot C = 0$, is satisfied on $\mathcal{U}_C \subset M$, then $R \cdot R = L_C \, Q(g, R)$, resp., $R \cdot R = 0$, holds on $\mathcal{U}_C$ ([37], resp., [58]). An example of a 4-dimensional Riemannian manifold satisfying $R \cdot C = 0$, with non-zero tensor $R \cdot R$, was found by Derdziński [18]. An example of a 4-dimensional submanifold isometrically immersed in a 6-dimensional Euclidean space $\mathbb{E}^6$ satisfying $R \cdot C = 0$, with non-zero tensor $R \cdot R$, was found by Zafindratafa [87]. For further results on 4-dimensional semi-Riemannian manifolds satisfying $R \cdot C = 0$ or $R \cdot C = L \, Q(g, C)$ we refer to the following papers: [20, 21, 40, 52]. We also refer to [3, 9, 26, 39, 59, 60, 62, 63, 77, 82, 83, 84, 85, 86]. for further results on semi-Riemannian manifolds satisfying (2), (3) or (4).

A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be a *quasi-Einstein manifold* if

$$\operatorname{rank}(S - \alpha \, g) = 1$$

on $\mathcal{U}_S \subset M$, where α is some function on this set (see, e.g., [28]). According to [23], *a Riemannian manifold M^n whose Ricci tensor has an eigenvalue of multiplicity at least n-1 is called quasi-Einstein.* Every non-Einstein warped product manifold $\overline{M} \times_F \widetilde{N}$ of a 1-dimensional $(\overline{M}, \overline{g})$ base manifold and a 2-dimensional manifold $(\widetilde{N}, \widetilde{g})$ or an $(n-1)$-dimensional Einstein manifold $(\widetilde{N}, \widetilde{g})$, $n \geq 4$, with a warping function F, is a quasi-Einstein manifold (see, e.g., [12, 29]).

Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations and the investigation on quasi-umbilical hypersurfaces of conformally flat spaces, see, e.g., [26] and references therein. Quasi-Einstein hypersurfaces in semi-Riemannian spaces of constant curvature were studied among other things in: [28, 45, 55].

According to a well-known Theorem of Cartan and Schouten, a hypersurface M in a conformally flat Riemannian manifold $\widetilde{N}$, $\dim \widetilde{N} \geq 5$, is conformally flat if and only if it is quasi-umbilical [5, 76]. This result remains valid when M is a conformally flat hypersurface in a conformally flat semi-Riemannian manifold $\widetilde{N}$, $\dim \widetilde{N} \geq 5$, [49]. From the above presented results we obtain immediately

Theorem 2.3 Every conformally flat hypersurface M isometrically immersed in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, is a quasi-Einstein manifold.

Further we have

Theorem 2.4 [17, 38] Let (M, g) be a 3-dimensional semi-Riemannian manifold or a conformally flat semi-Riemannian manifold of dimension ≥ 4. Then on $\mathcal{U}_S \subset M$ the following three conditions are equivalent to each other:

$$\begin{aligned} R \cdot R &= \rho\, Q(g, R), \\ R \cdot S &= \rho\, Q(g, S), \\ S^2 - \frac{tr(S^2)}{n}\, g &= \left(\frac{\kappa}{n-1} + (n-2)\rho \right) (S - \frac{\kappa}{n}\, g), \end{aligned}$$

where ρ is some function on $\mathcal{U}_S$.

Theorem 2.5 [39] Every conformally flat hypersurface M isometrically immersed in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, is a quasi-Einstein pseudosymmetric manifold. Precisely, if $\mathrm{rank}(S - \alpha\, g) = 1$ on $\mathcal{U}_S \subset M$ then

$$R \cdot R = \left(\frac{\kappa}{n-1} - \alpha \right) Q(g, R)$$

on $\mathcal{U}_S$, where α is some function on this set.

From this it follows immediately

Corollary 2.5 Let M be hypersurface isometrically immersed in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 3$. Let $\rho_1, \rho_2, \ldots, \rho_n$ be the eigenvalues of the Ricci operator $\mathcal{S}$ of M. If at every point of $\mathcal{U}_S \subset M$ we have $\rho_1 = \ldots = \rho_{n-1} \neq \rho_n$ then

$$\mathrm{rank}\, (S - \rho_1\, g) = 1 \quad \text{and} \quad R \cdot R = \frac{\rho_n}{n-1}\, Q(g, R)$$

on $\mathcal{U}_S$. We mention that 3-dimensional Riemannian manifolds with two disinct eigenvalues of the Ricci operator, i.e., with two distinct principal Ricci curvatures, were investigated among other things in: [61, 65, 66, 67, 68, 69, 70].

3. Roter spaces

Theorem 3.1 [53, Theorem 3.1, Theorem 3.2 (ii)] If (M, g), $n \geq 4$, is a semi-Riemannian manifold satisfying on the set $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ the following two conditions:

$$R \cdot R = L_R\, Q(g, R) \quad \text{and} \quad C \cdot C = L_C\, Q(g, C),$$

where L_R and L_C are some functions on $\mathcal{U}_S \cap \mathcal{U}_C$, then

$$R = \frac{\phi}{2}\, S \wedge S + \mu\, g \wedge S + \frac{\eta}{2}\, g \wedge g \tag{5}$$

on the set $\mathcal{U}$ of all points of $\mathcal{U}_S \cap \mathcal{U}_C$ at which $\mathrm{rank}\,(S - \tau\, g) > 1$ for any $\tau \in \mathbb{R}$, where $\phi,\ \mu,\ \eta$ are some functions defined on $\mathcal{U}$.

Theorem 3.2 [41, Theorem 3.1, Theorem 3.2 (ii)] If (M, g), $n \geq 4$, is a semi-Riemannian manifold satisfying on the set $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ the following two conditions:

$$R \cdot R = L_R\, Q(g, R) \quad \text{and} \quad R \cdot R - Q(S, R) = L\, Q(g, C),$$

where L_R and L are some functions on $\mathcal{U}_S \cap \mathcal{U}_C$, then (5) holds on the set $\mathcal{U}$ of all points of $\mathcal{U}_S \cap \mathcal{U}_C$ at which $\mathrm{rank}\,(S - \tau\, g) > 1$ for any $\tau \in \mathbb{R}$.

Theorem 3.3 (cf. [25, Proposition 3.2, Theorem 3.3, Theorem 4.4]) If (M, g), $n \geq 4$, is a semi-Riemannian manifold satisfying on the set $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ the following three conditions:

$$\begin{aligned} C \cdot C &= L_C\, Q(g, C), \\ R \cdot R - Q(S, R) &= L\, Q(g, C), \\ R \cdot S &= Q(g, D), \end{aligned}$$

where L and L_C are some functions on $\mathcal{U}_S \cap \mathcal{U}_C$ and D is a symmetric $(0, 2)$-tensor on this set, then the Roter equation (5) holds on the set $\mathcal{U}$ of all points of $\mathcal{U}_S \cap \mathcal{U}_C$ at which $\mathrm{rank}\,(S - \tau\, g) > 1$ for any $\tau \in \mathbb{R}$.

Theorem 3.4 (see, e.g., [32, 64]) Let (M, g), $n \geq 4$, be a semi-Riemannian manifold and let (5) be satisfied on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$, i.e.,

$$R \;=\; \frac{\phi}{2}\, S \wedge S + \mu\, g \wedge S + \frac{\eta}{2}\, g \wedge g\,,$$

where ϕ, μ and η are some functions on this set. Then on $\mathcal{U}_S \cap \mathcal{U}_C$ we have

$$\begin{aligned} S^2 &= \alpha_1\, S + \alpha_2\, g, \ \ \alpha_1 \;=\; \kappa + \frac{(n-2)\mu - 1}{\phi}, \ \ \alpha_2 \;=\; \frac{\mu\kappa + (n-1)\eta}{\phi}, \\ R \cdot R &= L_R\, Q(g, R), \ \ \ L_R \;=\; \frac{1}{\phi}\left((n-2)(\mu^2 - \phi\eta) - \mu\right), \\ R \cdot R &= Q(S, R) + L\, Q(g, C), \ \ \ L \;=\; L_R + \frac{\mu}{\phi} \;=\; \frac{n-2}{\phi}(\mu^2 - \phi\eta), \\ C \cdot R &= L_C\, Q(g, R), \ \ \ L_C \;=\; L_R + \frac{1}{n-2}\left(\frac{\kappa}{n-1} - \alpha_1\right), \\ C \cdot C &= L_C\, Q(g, C). \end{aligned}$$

Moreover, we have on $\mathcal{U}_S \cap \mathcal{U}_C$

$$\begin{aligned} R \cdot C &= L_R\, Q(g, C), \\ C \cdot R &= Q(S, C) + \left(L_R - \frac{\kappa}{n-1}\right) Q(g, C), \\ R \cdot C - C \cdot R &= \left(\frac{1}{\phi}(\mu - \frac{1}{n-2}) + \frac{\kappa}{n-1}\right) Q(g, R) \\ &\quad + \left(\frac{\mu}{\phi}(\mu - \frac{1}{n-2}) - \eta\right) Q(S, G), \\ R \cdot C - C \cdot R &= \frac{1}{n-2}\, Q(S, R) + \left(\frac{(n-1)\mu - 1}{(n-2)\phi} + \frac{\kappa}{n-1}\right) Q(g, R) \\ &\quad + \frac{\mu((n-1)\mu - 1) - (n-1)\phi\eta}{(n-2)\phi}\, Q(S, G), \end{aligned}$$

$$\begin{aligned} C \cdot R + R \cdot C &= Q(S,C) + \left(2L_R - \frac{\kappa}{n-1} \right) Q(g,C), \\ C \cdot R - R \cdot C &= Q(S,C) - \frac{\kappa}{n-1} Q(g,C). \end{aligned}$$

Remark 3.5 (i) We note that if (5) holds at a point of $\mathcal{U}_S \cap \mathcal{U}_C \subset M$, $n \geq 4$, then at this point we have $\mathrm{rank}(S - \alpha\, g) > 1$ for any $\alpha \in \mathbb{R}$ (see, e.g., [32, 43]).
(ii) (see, e.g., [32, 43, 64]) A semi-Riemannian manifold (M,g), $n \geq 4$, satisfying (5) on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ is called a *Roter space*, or a *Roter type space*, or a *Roter type manifold*.
(iii) (see, e.g., [78]) In the standard Schwarzschild coordinates $(t; r; \theta; \phi)$, and the physical units ($c = G = 1$), the Reissner-Nordström-de Sitter ($\Lambda > 0$), and the Reissner-Nordström-anti-de Sitter ($\Lambda < 0$) metrics are given by the line element

$$\begin{aligned} ds^2 &= -h(r)\, dt^2 + h(r)^{-1}\, dr^2 + r^2\, (d\theta^2 + \sin^2\theta\, d\phi^2), \\ h(r) &= 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}, \quad M, Q, \Lambda - \text{non-zero constants}. \end{aligned} \tag{6}$$

(iv) (see, e.g., [35, Section 2] The metric (6) satisfies (5) with

$$\begin{aligned} \phi &= \frac{3}{2}(Q^2 - Mr) r^4 Q^{-4}, \quad \mu = \frac{1}{2}(Q^4 + 3Q^2\Lambda r^4 - 3\Lambda M r^5) Q^{-4}, \\ \eta &= \frac{1}{12}(3Q^6 + 4Q^4\Lambda r^4 - 3Q^4 Mr + 9Q^2\Lambda^2 r^8 - 9\Lambda^2 M r^9) r^{-4} Q^{-4}. \end{aligned}$$

(v) If we set $\Lambda = 0$ in (6) then we obtain the line element of the Reissner-Nordström spacetime, see, e.g., [56, Section 9.2] and references therein. It seems that the Reissner-Nordström spacetime is the "oldest" known example of a Roter type warped product manifold.
(vi) [40, Abstract] *We determine a particular class of Roter type warped product manifolds. We show that every manifold of that class admits a non-trivial geodesic mapping onto some Roter type warped product manifold. Moreover, both geodesically related manifolds are pseudosymmetric of constant type.*
(vii) Some comments on pseudosymmetric manifolds (also called Deszcz symmetric spaces), as well as Roter spaces, are given in [14]: "From a geometric point of view, the Deszcz symmetric spaces may well be considered to be the simplest Riemannian manifolds next to the real space forms." and "From an algebraic point of view, Roter spaces may well be considered to be the simplest Riemannian manifolds next to the real space forms."
(viii) [13, Abstract] An algebraic classification of the Roter type spacetimes is given. It follows that the Roter-type curvature condition is essentially equivalent with the pseudosymmetry condition on 4-dimensional Lorentzian manifolds.
(ix) We mention that in the Chen's survey paper on Wintgen ideal submanifolds [Chen-2021], in Section 15 (Symmetry of Wintgen ideal submanifolds) results on Wintgen ideal submanifolds satisfying pseudosymmetry type curvature conditions are given. Among other things, the following result of [14] is presented (cf. [10, Theorem 15.11]: Let M^n ($n \geq 4$) be a Wintgen ideal submanifold of a real space form $R^m(c)$. Then M^n is pseudosymmetric if and only if M^n is a Roter space.

Example 3.6 [46, Example 4.1] (i) Let $S^p(\frac{1}{\sqrt{c_1}})$, be the p-dimensional, $p \geq 2$, standard sphere of radius $\frac{1}{\sqrt{c_1}}$, $c_1 = const. > 0$, with the standard metric $\overline{g}$. Let f be a non-constant function on $S^p(\frac{1}{\sqrt{c_1}})$ satisfying the following differential equation [72]

$$\overline{\nabla}(df) + c_1\, f\, \overline{g} = 0\,.$$

We set $F = (f + c)^2$, where c is a non-zero constant such that $f + c$ is either positive or negative on $S^p(\frac{1}{\sqrt{c_1}})$.
(ii) Let $(\widetilde{N}, \widetilde{g})$, $n - p = \dim \widetilde{N} \geq 2$, be a semi-Riemannian space of constant curvature c_2. We consider the warped product $S^p(\frac{1}{\sqrt{c_1}}) \times_F \widetilde{N}$ of the manifolds $S^p(\frac{1}{\sqrt{c_1}})$ and $(\widetilde{N}, \widetilde{g})$ with the above defined warping function F.
(iii) We can check that the warped product

$$S^p\left(\frac{1}{\sqrt{c_1}}\right) \times_F \widetilde{N}$$

satisfies the Roter equation. In particular, the warped product

$$S^p\left(\frac{1}{\sqrt{c_1}}\right) \times_F S^{n-p}\left(\frac{1}{\sqrt{c_2}}\right),$$

where $2 \leq p \leq n-2$ and $c_1 > 0,\ c_2 > 0$, also satisfies the Roter equation.
(iv) We also can prove that $S^p(\frac{1}{\sqrt{c_1}}) \times_F \widetilde{N}$ can be locally realized as a hypersurface isometrically immersed in a semi-Riemannian space of constant curvature.

Example 3.7 [31, Example 5.4] (i) Let $N^p_{s_1}(c_1) \times N^{n-p}_{s_2}(c_2)$ be the Cartesian product of two semi-Riemannian spaces of constant curvature $N^p_{s_1}(c_1)$ and $N^{n-p}_{s_2}(c_2)$, $2 \leq p \leq n-2$, where $c_1 = \frac{\kappa_1}{(p-1)p}$, $c_2 = \frac{\kappa_2}{(n-p-1)(n-p)}$ and κ_1 and κ_2 are the scalar curvatures of $N^p_{s_1}(c_1)$ and $N^{n-p}_{s_2}(c_2)$, respectively. The scalar curvature κ of the product $N^p_{s_1}(c_1) \times N^{n-p}_{s_2}(c_2)$ is expressed by

$$\kappa = \kappa_1 + \kappa_2 = p(p-1)\,c_1 + (n-p)(n-p-1)\,c_2.$$

The product $N^p_{s_1}(c_1) \times N^{n-p}_{s_2}(c_2)$ is a semisymmetric manifold ($R \cdot R = 0$). Moreover, if $c_1 + c_2 \neq 0$ then the following condition is satisfied

$$C \cdot C = -\frac{(p-1)(n-p-1)}{(n-2)(n-1)}\,(c_1 + c_2)\,Q(g, C).$$

(ii) We assume that c_1 and c_2 satisfy

$$(a)\ \ c_1 + c_2 \neq 0 \quad \text{and} \quad (b)\ \ (p-1)\,c_1 - (n-p-1)\,c_2 \neq 0.$$

Now we have $\mathcal{U}_S \cap \mathcal{U}_C = N^p_{s_1}(c_1) \times N^{n-p}_{s_2}(c_2)$. Thus the product $N^p_{s_1}(c_1) \times N^{n-p}_{s_2}(c_2)$ is a non-conformally flat and non-Einstein semi-Riemannian manifold. Moreover

$$\begin{aligned}
\phi &= \tau\,(c_1 + c_2),\\
\mu &= -(n-2)\tau\, c_1 c_2,\\
\eta &= \tau\, c_1 c_2\,((p-1)^2 c_1 + (n-p-1)^2 c_2),\\
\tau &= ((p-1)c_1 - (n-p-1)c_2)^{-2}.
\end{aligned}$$

Remark 3.8 (i) Let S and $\mathcal{S}$ be the Ricci tensor and the Ricci operator of a semi-Riemannian manifold (M, g), $n \geq 3$, respectively. The $(0,2)$-tensor S^2 of (M, g) is defined by $S^2(X, Y) = S(\mathcal{S}(X), Y)$, where X and Y are vector fields on M.
(ii) It is easy to verify that the following identity is satisfied on every Einstein semi-Riemannian manifold (M, g), $n \geq 4$,

$$g \wedge S^2 + \frac{n-2}{2}\,S \wedge S - \kappa\, g \wedge S + \frac{\kappa^2 - \mathrm{tr}_g(S^2)}{2(n-1)}\, g \wedge g \;=\; 0. \tag{7}$$

(iii) [27, Lemma 2.1] If $\operatorname{rank}(S - \alpha\, g) = 1$ on $\mathcal{U}_S \subset M$, where α is some function on $\mathcal{U}_S$, then (7) holds on $\mathcal{U}_S$.
(iv) [27, Lemma 2.2] If (M, g), $n \geq 4$, is a semi-Riemannian manifold satisfying (5), i.e.,

$$R \;=\; \frac{\phi}{2}\, S \wedge S + \mu\, g \wedge S + \frac{\eta}{2}\, g \wedge g$$

on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then

$$C \;=\; \frac{\phi}{n-2}\left(g \wedge S^2 + \frac{n-2}{2}\, S \wedge S - \kappa\, g \wedge S + \frac{\kappa^2 - \mathrm{tr}_g(S^2)}{2(n-1)}\, g \wedge g \right)$$

on this set.

4. The difference tensor $R \cdot C - C \cdot R$

On every semi-Riemannian manifold (M, g), $n \geq 4$, the following identity is satisfied

$$(n-2)(R \cdot C - C \cdot R) = Q(S, R) - \frac{\kappa}{n-1}\, Q(g, R) - g \wedge (R \cdot S) + P, \tag{8}$$

where the $(0, 6)$-tensor P is defined by

$$\begin{aligned} & P(X_1, X_2, X_3, X_4; X, Y) \\ = \;& g(X, X_1)\, R(\mathcal{S}(Y), X_2, X_3, X_4) - g(Y, X_1)\, R(\mathcal{S}(X), X_2, X_3, X_4) \\ & + g(X, X_2) R(X_1, \mathcal{S}(Y), X_3, X_4) - g(Y, X_2)\, R(X_1, \mathcal{S}(X), X_3, X_4) \\ & + g(X, X_3)\, R(X_1, X_2, \mathcal{S}(Y), X_4) - g(Y, X_3)\, R(X_1, X_2, \mathcal{S}(X), X_4) \\ & + g(X, X_4)\, R(X_1, X_2, X_3, \mathcal{S}(Y)) - g(Y, X_4)\, R(X_1, X_2, X_3, \mathcal{S}(X)). \end{aligned}$$

The local expression of the basic identity (8) is the following

$$\begin{aligned} & (n-2)(R \cdot C - C \cdot R)_{hijklm} \\ = \;& Q(S, R)_{hijklm} - \frac{\kappa}{n-1}\, Q(g, R)_{hijklm} \\ & + g_{hl} A_{mijk} - g_{hm} A_{lijk} - g_{il} A_{mhjk} + g_{im} A_{lhjk} \\ & + g_{jl} A_{mkhi} - g_{jm} A_{lkhi} - g_{kl} A_{mjhi} + g_{km} A_{ljhi} \\ & - g_{ij} (R \cdot S)_{hklm} - g_{hk} (R \cdot S)_{ijlm} + g_{ik} (R \cdot S)_{hjlm} + g_{hj} (R \cdot S)_{iklm}, \end{aligned}$$

where $A_{mijk} \;=\; g^{rs} S_{mr} R_{sijk}$ and

$$\begin{aligned} (R \cdot S)_{hklm} &= g^{rs} (S_{hr} R_{sklm} + S_{kr} R_{shlm}), \\ Q(g, S)_{hklm} &= g_{hl} S_{km} + g_{kl} S_{hm} - g_{hm} S_{kl} - g_{km} S_{hl}, \\ Q(g, R)_{hijklm} &= g_{hl} R_{mijk} + g_{il} R_{hmjk} + g_{jl} R_{himk} + g_{kl} R_{hijm} \\ & \quad - g_{hm} R_{lijk} - g_{im} R_{hljk} - g_{jm} R_{hilk} - g_{km} R_{hijl}, \\ Q(S, R)_{hijklm} &= S_{hl} R_{mijk} + S_{il} R_{hmjk} + S_{jl} R_{himk} + S_{kl} R_{hijm} \\ & \quad - S_{hm} R_{lijk} - S_{im} R_{hljk} - S_{jm} R_{hilk} - S_{km} R_{hijl}, \\ (R \cdot C)_{hijklm} &= g^{rs} (C_{rijk} R_{shlm} + C_{hrjk} R_{silm} + C_{hirk} R_{sjlm} + C_{hijr} R_{sklm}), \\ (C \cdot R)_{hijklm} &= g^{rs} (R_{rijk} C_{shlm} + R_{hrjk} C_{silm} + R_{hirk} C_{sjlm} + R_{hijr} C_{sklm}). \end{aligned}$$

Theorem 4.1 [44, Theorem 3.1] On every Einstein semi-Riemannian manifold (M, g), $n \geq 4$, the following identities are satisfied

$$R \cdot C - C \cdot R = \frac{\kappa}{(n-1)n}\, Q(g, R) = \frac{\kappa}{(n-1)n}\, Q(g, C). \tag{9}$$

Remark 4.2 Let (M, g), $n \geq 4$, be an Einstein semi-Riemannian manifold. From (9) we get immediately

$$R \cdot C - C \cdot R = \frac{1}{n-1}\, Q(S, R) = \frac{1}{n-1}\, Q(S, C).$$

Moreover, on every Ricci flat semi-Riemannian manifold we have

$$R \cdot C = C \cdot R.$$

Theorem 4.3 [35, Theorem 2.3] Let (M, g), $n \geq 4$, be an Einstein semi-Riemannian manifold.
(i) The condition (1), i.e.,

$$R \cdot C - C \cdot R \;=\; \frac{\kappa}{n-1}\, Q(g, C) - Q(S, C)$$

on M.
(ii) If the condition $R \cdot R = L_R Q(g, R)$ is satisfied on $U_R \subset M$ then on this set we have

$$\begin{aligned} R \cdot R - Q(S, R) &= \left(L_R - \frac{\kappa}{n}\right) Q(g, C), \\ C \cdot C &= \left(L_R - \frac{\kappa}{(n-1)n}\right) Q(g, C), \\ R \cdot C + C \cdot R &= Q(S, C) + \left(2L_R - \frac{\kappa}{n-1}\right) Q(g, C). \end{aligned}$$

Theorem 4.4 ([44, Theorem 4.1, Proposition 4.2, Proposition 4.3], [28, Theorem 6.4]) Let (M, g), $n \geq 4$, be a semi-Riemannian manifold.
(i) If the condition

$$R \cdot C - C \cdot R = L\, Q(g, C)$$

is satisfied on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then

$$R \cdot R = L\, Q(g, R), \quad C \cdot R = 0$$

on this set.
(ii) If the conditions

$$S = \mu\, g + \beta\, w \otimes w, \quad \sum_{X,Y,Z} w(X)\, \mathcal{C}(Y, Z) = 0$$

are satisfied on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$, where μ and β are some functions and w a covector field on this set, then

$$R \cdot R = \frac{\kappa}{(n-1)n}\, Q(g, R), \quad C \cdot R = 0$$

on $\mathcal{U}_S \cap \mathcal{U}_C$. Consequently on this set we have

$$R \cdot C - C \cdot R = \frac{\kappa}{(n-1)n}\, Q(g, C).$$

(iii) If the condition

$$C = \frac{\lambda}{2}\, (S - \alpha\, g) \wedge (S - \alpha\, g)$$

is satisfied on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$, where α and λ are some functions on $\mathcal{U}_S \cap \mathcal{U}_C$, then

$$C \cdot R = 0, \quad R \cdot R = \left(\frac{\kappa}{n-1} - \lambda \right) Q(g, R)$$

on $\mathcal{U}_S \cap \mathcal{U}_C$. Consequently on this set we have

$$R \cdot C - C \cdot R = \left(\frac{\kappa}{n-1} - \lambda \right) Q(g, C).$$

Theorem 4.5 ([42, Theorem 3.1], [28, Theorem 6.1]) Let (M, g), $n \geq 4$, be a semi-Riemannian manifold satisfying on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$

$$R \cdot C - C \cdot R = L\, Q(g, R), \quad S - \alpha\, g = \beta\, w \otimes w,$$

where w is some 1-form on $\mathcal{U}_S \cap \mathcal{U}_C$ and L, α, β are some functions on $\mathcal{U}_S \cap \mathcal{U}_C$. Then on this set we have:

$$S = \beta\, w \otimes w, \quad \kappa = 0, \quad \sum_{X,Y,Z} w(X)\, \mathcal{C}(Y, Z) = 0, \quad R \cdot R = 0, \quad R \cdot C = C \cdot R = 0.$$

Theorem 4.6 ([42, Proposition 3.2], [28, Theorem 6.1 (ii)]) Let (M, g), $n \geq 4$, be a semi-Riemannian manifold satisfying on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$

$$S = \beta\, w \otimes w, \quad \kappa = 0, \quad \sum_{X,Y,Z} w(X)\mathcal{C}(Y, Z) = 0.$$

Then on this set we have $R \cdot R = 0$, $C \cdot R = 0$, and consequently

$$R \cdot C = C \cdot R = 0.$$

Theorem 4.7 ([42, Theorem 4.1], [28, Theorem 6.2 (i)]) Let (M, g), $n \geq 4$, be a semi-Riemannian manifold satisfying on $\mathcal{U} = \mathcal{U}_S \cap \mathcal{U}_C \subset M$

$$R \cdot C - C \cdot R = L\, Q(g, R),$$

where L is some function on $\mathcal{U}$. Moreover, let at every point of $\mathcal{U}$: rank $(S - \tau\, g) > 1$ for any $\tau \in \mathbb{R}$. Then on this set we have

$$R = \frac{\phi}{2}\, S \wedge S + \mu\, g \wedge S + \frac{\eta}{2}\, g \wedge g,$$

where ϕ, μ, η are some functions on $\mathcal{U}$ satisfying

$$\eta = \alpha\mu, \quad \alpha = \frac{(n-2)\mu - 1}{(n-2)\phi}.$$

Consequently, on $\mathcal{U}$ we have

$$R \cdot R = 0 \quad \text{and} \quad R \cdot C = 0.$$

We have also the following inverse statement.

Theorem 4.8 ([42, Proposition 4.1], [28, Theorem 6.2 (ii)]) Let (M, g), $n \geq 4$, be a semi-Riemannian manifold. If on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ the tensor R is of the form

$$R \;=\; \frac{\phi}{2}\, S \wedge S + \mu\, g \wedge S + \frac{\eta}{2}\, g \wedge g,$$

with the functions ϕ, μ, η satisfying on this set

$$\eta = \alpha\mu, \quad \alpha = \frac{(n-2)\mu - 1}{(n-2)\phi}$$

then

$$R \cdot R = 0, \quad C \cdot R = -L_2\, Q(g,R), \quad L_2 = \alpha + \frac{\kappa}{n-1}, \quad R \cdot C - C \cdot R = L_2\, Q(g,R).$$

Remark 4.9 Let (M,g), $n \geq 4$, be a semi-Riemannian manifold. If the Riemann-Christoffel curvature tensor R has on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ the form (5), i.e.,

$$R = \frac{\phi}{2} S \wedge S + \mu\, g \wedge S + \eta\, G,$$

with some functions ϕ, μ, η on $\mathcal{U}_S \cap \mathcal{U}_C$, then on this set we have

$$\begin{aligned} R \cdot R &= L_R\, Q(g,R), \quad L_R = (n-2)(\alpha\mu - \eta), \\ C \cdot R &= L_C\, Q(g,R), \quad L_C = L_R - \frac{\kappa}{n-1} - \alpha, \\ R \cdot C - C \cdot R &= \left(\alpha + \frac{\kappa}{n-1}\right) Q(g,R) + (\alpha\mu - \eta)\, Q(S,G), \\ R \cdot C - C \cdot R &= (L_R - L_C)\, Q(g,R) + \frac{1}{n-2} L_R\, Q(S,G), \end{aligned}$$

where

$$\alpha = \frac{(n-2)\mu - 1}{(n-2)\phi} \quad \text{and} \quad G = \frac{1}{2}\, g \wedge g.$$

The last equation, by the identity

$$Q(S,G) = -Q(g, g \wedge S),$$

turns into

$$\begin{aligned} R \cdot C - C \cdot R &= (L_R - L_C)\, Q(g,R) - \frac{1}{n-2} L_R\, Q(g, g \wedge S) \\ &= Q(g,\ (L_R - L_C)\, R - \frac{1}{n-2} L_R\, g \wedge S). \end{aligned}$$

We recall that

$$R \cdot C - C \cdot R = -Q(S,C) + \frac{\kappa}{n-1}\, Q(g,C) = Q\left(\frac{\kappa}{n-1}\, g - S, C\right)$$

on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$.

Remark 4.10 (i) We refer to: [6, 7, 8, 9, 10, 73] for fundamental results on Chen ideal submanifolds.
(ii) Chen ideal submanifolds satisfying pseudosymmetry type curvature conditions were investigated among others in [33, 47].
(iii) Chen ideal submanifolds satisfying some generalized Einstein metric curvature conditions were studied in [48]. For instance, Theorems 5, 6 and 7 of [48] contain results on Chen ideal submanifolds M in $\mathbb{E}^{n+m}$, $n \geq 4$, $m \geq 1$, satisfying at every point of $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ one of the following conditions:

(a) the tensor $R \cdot C - C \cdot R$ and the tensors $Q(g, R)$ and $Q(S, R)$ are linearly dependent,
(b) the tensor $R \cdot C - C \cdot R$ and the tensors $Q(g, C)$ and $Q(S, C)$ are linearly dependent,
(c) the tensor $R \cdot C - C \cdot R$ and the tensors $Q(g, g \wedge S)$ and $Q(S, g \wedge S)$ are linearly dependent.

Generalized Robertson-Walker spacetimes also satisfies generalized Einstein metric curvature conditions. For instance, we have

Theorem 4.11 [12, Theorem 4.2] The warped product $\overline{M} \times_F \widetilde{N}$ of an 1-dimensional manifold $(\overline{M}, \overline{g})$ and an $(n-1)$-dimensional, $n \geq 5$, Einstein manifold $(\widetilde{N}, \widetilde{g})$, which is not of constant curvature, satisfies

$$\begin{aligned} \operatorname{rank}\left(S - \left(\frac{\kappa}{n-1} - L_S\right) g\right) &= 1, \\ R \cdot S &= L_S\, Q(g, S), \\ (n-2)\,(R \cdot C - C \cdot R) &= Q(S, R) - L_S\, Q(g, R), \end{aligned}$$

where L_S is some function on $\overline{M} \times_F \widetilde{N}$.

Corollary 4.12 The warped product $\overline{M} \times_F \widetilde{N}$ of an 1-dimensional manifold $(\overline{M}, \overline{g})$ and an $(n-1)$-dimensional, $n \geq 4$, Einstein manifold $(\widetilde{N}, \widetilde{g})$, which is not of constant curvature, with the warping function $F(t) = t^2$, satisfies

$$\begin{aligned} \operatorname{rank}\left(S - \frac{\kappa}{n-1}\, g\right) &= 1, \\ R \cdot S &= 0, \\ (n-2)\,(R \cdot C - C \cdot R) &= Q(S, R). \end{aligned}$$

Theorem 4.13 [2, Theorem 4.1] If the warped product $\overline{M} \times_F \widetilde{N}$ of an 1-dimensional manifold $(\overline{M}, \overline{g})$ and an $(n-1)$-dimensional, $n \geq 4$, non-Einstein manifold $(\widetilde{N}, \widetilde{g})$, satisfies on the set U, of all points of $\mathcal{U}_S \cap \mathcal{U}_C \subset \overline{M} \times_F \widetilde{N}$ at which $Q(S, R) \neq 0$, the condition

$$R \cdot C - C \cdot R \;=\; L\, Q(S, R),$$

for some function L on U, then

$$F(t) = (a\, t + b)^2, \quad a, b \in \mathbb{R}, \quad a \neq 0, \quad L = \frac{1}{n-2}.$$

We also recall the following result on 4-dimensional gneralized Robertson-Walker spacetimes.

Theorem 4.14 [16, Theorem 4.1] The warped product manifolds $\overline{M} \times_F \widetilde{N}$ of an 1-dimensional manifold $(\overline{M}, \overline{g})$ and a 3-dimensional semi-Riemannian manifold $(\widetilde{N}, \widetilde{g})$ satisfies

$$R \cdot R - Q(S, R) = L\, Q(g, C),$$

where L is some function.

5. Hypersurfaces in semi-Riemannian spaces of constant curvature

Let $N_s^{n+1}(c)$, $n \geq 3$, be a semi-Riemannian space of constant curvature $c = \frac{\widetilde{\kappa}}{n(n+1)}$ with signature $(s, n+1-s)$, where $\widetilde{\kappa}$ is its scalar curvature. Let M be a hypersurface isometrically immersed in $N_s^{n+1}(c)$ and let g be the metric tensor induced on M from the metric of the ambient space and R and κ the Riemann-Christoffel curvature tensor and the scalar curvature, respectively.

Let H and $\mathcal{A}$ be the second fundamental tensor and the shape operator of M, respectively. We have

$$H(X,Y) = g(\mathcal{A}X, Y),$$

for any vectors fields X, Y tangent to M. The $(0,2)$-tensors H^2 and H^3 are defined by

$$\begin{aligned} H^2(X,Y) &= H(\mathcal{A}X, Y), \\ H^3(X,Y) &= H^2(\mathcal{A}X, Y), \end{aligned}$$

respectively.

The Gauss equation of M in $N_s^{n+1}(c)$ reads

$$R = \frac{\varepsilon}{2}\, H \wedge H + \frac{\widetilde{\kappa}}{2n(n+1)}\, g \wedge g, \quad \varepsilon = \pm 1. \tag{10}$$

On every hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, we have [49]

$$R \cdot R = Q(S,R) - \frac{(n-2)\widetilde{\kappa}}{n(n+1)}\, Q(g,C). \tag{11}$$

Theorem 5.1 [51] Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 3$, satisfying on $\mathcal{U}_R \subset M$

$$\mathcal{A}^2 = \alpha\, \mathcal{A} + \beta\, Id \tag{12}$$

where α and β are some functions on this set and Id is the identity transformation of M. Then

$$R \cdot R = \left(\frac{\widetilde{\kappa}}{n(n+1)} - \varepsilon\beta \right) Q(g,R)$$

on $\mathcal{U}_R$.

Corollary 5.2 A hypersurface M in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 3$, having at every point of $\mathcal{U}_R \subset M$ two distinct principal curvatures is pseudosymmetric.

Let $\mathcal{U}_{\mathcal{A}} \subset M$ be the set of all points at which $\mathcal{A}^2$ cannot be expressed by a linear combination of the second fundamental tensor $\mathcal{A}$ and the identity transformation Id of M. We can prove that

$$\mathcal{U}_{\mathcal{A}} \subset \mathcal{U}_S \cap \mathcal{U}_C \subset M.$$

Moreover, at at every point of $(\mathcal{U}_S \cap \mathcal{U}_C) \setminus \mathcal{U}_{\mathcal{A}}$ we have

$$\mathcal{A}^2 = \alpha\, \mathcal{A} + \beta\, Id \quad \text{and} \quad \mathrm{tr}(\mathcal{A}) - \alpha \neq 0.$$

From the Gauss equation we have:

$$S = \varepsilon\, (\mathrm{tr}(\mathcal{A})\, H - H^2) + \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\, g,$$

which, by (12), turns into

$$S = \varepsilon\, (\mathrm{tr}(\mathcal{A}) - \alpha)\, H + \left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \varepsilon\beta \right) g.$$

Theorem 5.3 [54] Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 3$, satisfying (12) on $(\mathcal{U}_S \cap \mathcal{U}_C) \setminus \mathcal{U}_{\mathcal{A}}$, where α and β are some functions on this set. Then

$$R = \frac{\phi}{2}\, S \wedge S + \mu\, g \wedge S + \frac{\eta}{2}\, g \wedge g$$

on $(\mathcal{U}_S \cap \mathcal{U}_C) \setminus \mathcal{U}_{\mathcal{A}}$, where

$$\begin{aligned} \phi &= \varepsilon\,(\mathrm{tr}(H) - \alpha)^{-2}, \\ \mu &= \phi\left(\varepsilon\beta - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right), \\ \eta &= \phi\left(\varepsilon\beta - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right)^2 + \frac{\widetilde{\kappa}}{n(n+1)}. \end{aligned}$$

Corollary 5.4 [54, Corollary 3.1, Example 3.2] On every Clifford torus

$$S^p\left(\sqrt{\frac{p}{n}}\right) \times S^{n-p}\left(\sqrt{\frac{n-p}{n}}\right), \quad n \neq 2p, \quad 2 \leq p \leq n-2,$$

the following equation is satisfied

$$R = \frac{\phi}{2}\, S \wedge S + \mu\, g \wedge S + \frac{\eta}{2}\, g \wedge g.$$

Theorem 5.5 [22] A hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, is a pseudosymmetric manifold if and only if at every point of $\mathcal{U}_R \subset M$ we have:

$$\mathcal{A}^2 = \alpha\,\mathcal{A} + \beta\, Id, \ \ \alpha, \beta \in \mathbb{R} \quad \text{or} \quad \mathrm{rank}\, H = 2.$$

Theorem 5.6 [15] Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying on $\mathcal{U}_{\mathcal{A}} \subset M$ the condition rank $H = 2$. Then on this set we have

$$R \cdot R = \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g, R).$$

Theorem 5.7 [30, Theorem 6.5] If on the subset $\mathcal{U}_{\mathcal{A}}$ of a hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, one of the tensors $R \cdot C$, $C \cdot R$ or $R \cdot C - R \cdot C$ is a linear combination of $R \cdot R$ and of a finite sum of tensors of the form $Q(E, B)$, where E is a symmetric $(0, 2)$-tensor and B a generalized curvature tensor, then

$$\mathcal{A}^3 = \mathrm{tr}(\mathcal{A})\, \mathcal{A}^2 + \psi\, \mathcal{A} + \rho\, Id,$$

or equivalently,

$$H^3 = \mathrm{tr}(H)\, H^2 + \psi\, H + \rho\, g$$

on $\mathcal{U}_{\mathcal{A}}$, where ψ and ρ are some functions on this set.

Theorem 5.8 [34, Theorem 5.2 (iii)] Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying on $\mathcal{U}_{\mathcal{A}} \subset M$ the following equation

$$R \cdot C - C \cdot R = Q(g, B),$$

where B is a generalized curvature tensor. Then on $\mathcal{U}_{\mathcal{A}}$ we have

$$\begin{aligned} B = & \left(-\frac{\varepsilon\psi}{n-1} + \frac{\widetilde{\kappa}}{n(n+1)}\right) R + \left(-\frac{\varepsilon\psi}{n-1} + \frac{2\widetilde{\kappa}}{n(n+1)}\right) g \wedge S \\ & - \frac{1}{n-1}\, g \wedge S^2 - \frac{1}{2(n-2)(n-1)}\, S \wedge S + \lambda\, G, \end{aligned}$$

where λ is a function on this set.

Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying on $\mathcal{U}_{\mathcal{A}} \subset M$

$$\mathcal{A}^3 \;=\; \phi\,\mathcal{A}^2 + \psi\,\mathcal{A} + \rho\,Id, \tag{13}$$

or equivalently,

$$H^3 = \phi\,H^2 + \psi\,H + \rho\,g$$

on $\mathcal{U}_{\mathcal{A}}$, where ϕ, ψ and ρ are some functions on this set. We set

$$\begin{aligned}
\mu &= \frac{1}{n-2}\left(\frac{\kappa}{n-1} - \frac{\widetilde{\kappa}}{n+1}\right),\\
\beta_1 &= \varepsilon\,(\phi - tr(\mathcal{A})),\\
\beta_2 &= -\frac{\varepsilon}{n-2}\left(\phi\,(2tr(\mathcal{A}) - \phi) - (tr(\mathcal{A}))^2 - \psi - (n-2)\varepsilon\mu\right),\\
\beta_3 &= \varepsilon\mu tr(\mathcal{A}) + \frac{1}{n-2}\left(\psi\,(2tr(\mathcal{A}) - \phi) + (n-3)\rho\right),\\
\beta_4 &= \beta_3 - \varepsilon\beta_2 tr(\mathcal{A}) + \frac{(n-1)\widetilde{\kappa}\beta_1}{n(n+1)},\\
\beta_5 &= \frac{\kappa}{n-1} + \varepsilon\psi - \frac{(n^2-3n+3)\widetilde{\kappa}}{n(n+1)} + \beta_1 tr(\mathcal{A}),\\
\beta_6 &= \beta_2 - \frac{(n-3)\widetilde{\kappa}}{n(n+1)}.
\end{aligned} \tag{14}$$

Theorem 5.9 [75, Theorem 6.7] If M is a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying on $\mathcal{U}_{\mathcal{A}} \subset M$ equation (13) then on this set we have

$$\begin{aligned}
(n-2)\,R\cdot C &= \rho\,Q(H,G) - \frac{(n-2)^2\widetilde{\kappa}}{n(n+1)}\,Q(g,R)\\
&\quad +(n-2)\,Q(S,R) - \frac{(n-3)\widetilde{\kappa}}{n(n+1)}\,Q(S,G)\\
&\quad +(\phi - tr(\mathcal{A}))\,g\wedge Q(H,H^2),
\end{aligned}$$

$$\begin{aligned}
(n-2)\,C\cdot R &= \left(\frac{\kappa}{n-1} + \varepsilon\psi - \frac{(n^2-3n+3)\widetilde{\kappa}}{n(n+1)}\right) Q(g,R)\\
&\quad +(n-3)\,Q(S,R) - \frac{(n-3)\widetilde{\kappa}}{n(n+1)}\,Q(S,G)\\
&\quad +(\phi - tr(\mathcal{A}))\,H\wedge Q(g,H^2),
\end{aligned} \tag{15}$$

$$\begin{aligned}
(n-2)\,(R\cdot C - C\cdot R) &= Q(S,R) + \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\,Q(g,R) + \rho\,Q(H,G)\\
&\quad +(\phi - tr(\mathcal{A}))\,(g\wedge Q(H,H^2) - H\wedge Q(g,H^2)),
\end{aligned} \tag{16}$$

$$\begin{aligned}
(n-2)\,C\cdot C &= (n-3)\,Q(S,R) + \beta_1\,Q(S,g\wedge H) + \beta_4\,Q(H,G)\\
&\quad +\beta_5\,Q(g,R) + \beta_6\,Q(S,G),
\end{aligned} \tag{17}$$

$$R \cdot S \;=\; \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g,S) + \rho\, Q(g,H) - \varepsilon\beta_1\, Q(H,H^2), \tag{18}$$

where $\beta_1, \ldots, \beta_6$ are defined by (14). In particular, if M is a hypersurface in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, having at every point of $\mathcal{U}_{\mathcal{A}} \subset M$ exactly three distinct principal curvatures ρ_1, ρ_2 and ρ_3 then (16) - (18) hold on $\mathcal{U}_{\mathcal{A}}$ with

$$\begin{aligned} \phi &= \rho_1 + \rho_2 + \rho_3, \\ \psi &= -(\rho_1\rho_2 + \rho_1\rho_3 + \rho_2\rho_3), \\ \rho &= \rho_1\rho_2\rho_3. \end{aligned}$$

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SURFACES ASSOCIATED WITH PASCAL AND CATALAN TRIANGLES

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ABSTRACT

An open problem in reliability theory is that of calculating the coefficients of the reliability polynomials associated with particular networks. Since the reliability polynomials can be expressed in Bernstein form (related to binomial coefficients), an extension of the classical discrete Pascal's triangle to a continuous version (the *Pascal's Surface*) might be geometrically helpful and revealing [6]. Another famous triangle, deeply involved in the reliability theory and counting problems, is the Catalan triangle (formed by the ballot numbers [2]), which can be also extended to a continuous surface [5]. We have investigated some geometric properties of these two surfaces: Gauss curvature, mean curvature, geodesics and level curves, as well as their symmetries [1, 2].

Keywords Reliability theory, Catalan triangle, Pascal triangle.

1. Pascal's Triangle

Probably the most famous triangular arrangement of integers is the one containing the coefficients of the binomial expansion of $(x+y)^n$. It is known as *Pascal's triangle*, although the triangle itself has been known and studied many centuries earlier by other mathematicians in India (Acharya Pingala 3rd/2nd century B.C.), Halayudha (c. 10th century), the Arab world and Persia (Al-Karaji (953–1029) and Omar Khayyam (1048–1131)), China (Jia Xian (1010-1070), Yang Hui (1238-1298), and Zhu Shijie (1249-1314)), as well as Europe (Ramon Llull (1232-1316), Michael Stifel (1487-1567), Petrus Apianus (1495-1552), Niccolo Tartaglia (1499-1557) and Marin Mersenne (1588-1648)).

Still, Pascal proved several important properties of the binomial coefficients, and wrote the first modern treatise regarding this arithmetical triangle.

It seems that Blaise Pascal became aware of this arithmetical triangle for the first time when still in his teens, during a visit to Mersenne. In 1636, Father Mersenne published a large arithmetical triangle in *Harmonicorum Libri XII* (he wanted to apply the knowledge of combinatorics to musical theory). Almost two decades later, in 1654, Pascal wrote *Traité du Triangle Arithmétique*, which was not printed until 1665 (after Pascal's death). Among other aspects, it details 19 properties (Pascal called them Consequences) of the binomial coefficients that could be derived from this arithmetic triangle. Some of the most important identities are represented by Consequences V and VIII, which currently can be found in any high school math curriculum.

The first mathematician who named the triangle after Pascal was Pierre Raymond de Montmort in 1708, who called it *Table de M. Pascal pour les combinaisons.* Around 1730, Abraham de Moivre called it *Triangulum Arithmeticum Pascalianum* and this name has stuck with Western scientists, while being called Khayyam's triangle in Iran, and Yang Hui's triangle in China.

It should be mentioned that, using Pascal's triangle, one can derive and investigate many notable sequences of integers, e.g., Fibonacci, Catalan, Lucas, Bernoulli, and Stirling numbers.

Over time, Pascal's triangle has been represented in more than one form. In the following, we consider $P(\infty)$ the infinite symmetric matrix of components $P(\infty)_{i,j} = \binom{i+j}{i}$, for $i, j \geq 0$:

$$P(\infty) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ 1 & 3 & 6 & 10 & 15 & 21 & \dots \\ 1 & 4 & 10 & 20 & 35 & 56 & \dots \\ 1 & 5 & 15 & 35 & 70 & 126 & \dots \\ 1 & 6 & 21 & 56 & 126 & 252 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1}$$

$P(\infty)$ is a perfect match of the form described by Pascal! (see Figure 1).
(`https://commons.wikimedia.org/wiki/File:TrianguloPascal.jpg`)

Direct computation shows that

$$P(\infty) = L(\infty) \cdot L(\infty)^t, \tag{2}$$

where $L(\infty)$ is the infinite lower triangular matrix

$$L(\infty) = \begin{pmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ 1 & 3 & 3 & 1 & & & \\ 1 & 4 & 6 & 4 & 1 & & \\ 1 & 5 & 10 & 10 & 5 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{3}$$

with components $L(\infty)_{i,j} = \binom{i}{j}$, where $\binom{i}{j} = 0$ if $i < j$.

Using the Euler's function gamma, which is a natural extension of the factorial to real numbers,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad \Gamma(n+1) = n!, \quad n = 0, 1, \dots,$$

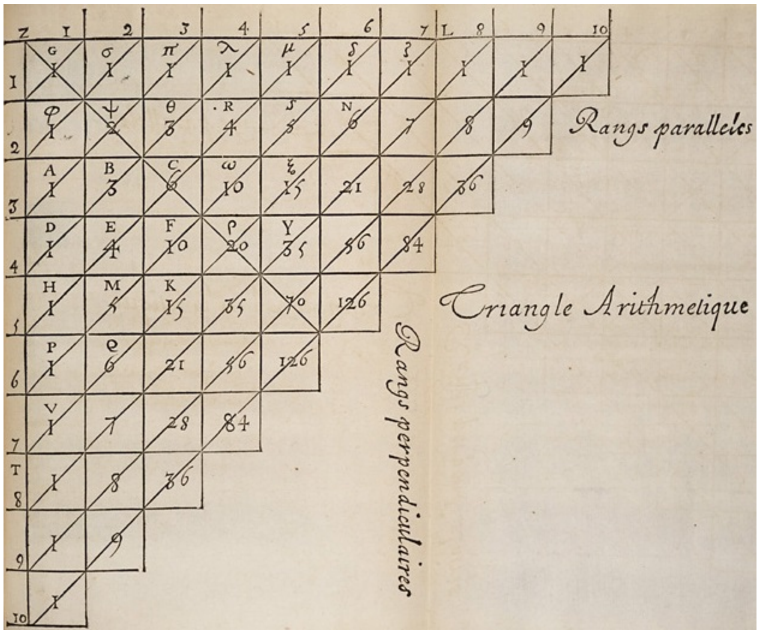


Figure 1:

Fowler [4] considered the extension of the classical integer binomial coefficients $\binom{n}{k}$ to real numbers $\binom{y}{x}$ (for $0 \leq x \leq y$); he mentioned that this surface defined by the binomial function $C = y!/x!(y-x)!$ is Pascal's triangle *interpolated to a steeply rising ridge* and that he knows *no evidence that the graph of C has ever been plotted before*. This corresponds to the $L(\infty)$ form of the triangle.

Pellicer and Alvo [7] introduced a generalization of the Pascal's triangle (the *Modified Pascal Triangle*) and extended the discrete Pascal's triangle (as well as its modified version) to a continuous graphical model corresponding to $P(\infty)$. They called these surfaces *Pascal Surfaces* and constructed their equation using the Euler's functions Gamma and Beta.

2. A Continuous Pascal's Surface

Let xOy be a Cartesian coordinate system. Starting from the original Pascal's triangle, we consider the 3-dimensional version of the infinite triangle corresponding to $P(\infty)$ formed by the points $\left(i, j, \binom{i+j}{i}\right)$, for every nonnegative integers i, j (see Figure 2).

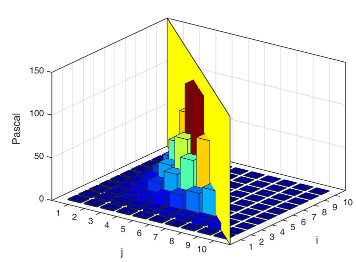


Figure 2: The 3-dimensional representation of the Pascal's Triangle

A continuous smooth surface interpolating the points $\left(i, j, \binom{i+j}{i}\right)$ is simply obtained by replacing the binomial coefficients $\binom{x+y}{y} = \frac{(x+y)!}{x!y!}$ with the continuous version $\frac{\Gamma(x+y+1)}{\Gamma(x+1)\Gamma(y+1)}$. Thus, we obtain the *Pascal's Surface* defined by the function

$$f : \mathbb{R}_+^2 \to \mathbb{E}^3,\ f(x,y) = (x, y, z(x,y))\,, wherez(x,y) = \frac{\Gamma(x+y+1)}{\Gamma(x+1)\Gamma(y+1)}, \quad x, y \geq 0.$$

We have studied the geometrical properties of this surface (presented in Figure 3). Recall that

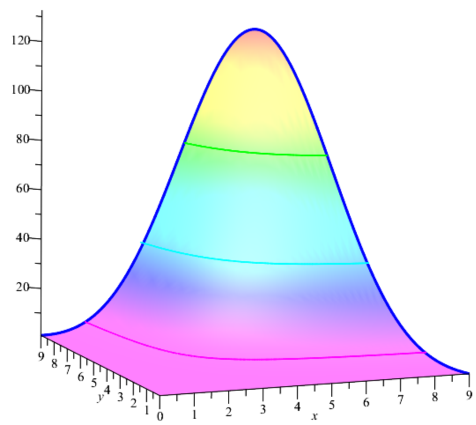


Figure 3: Pascal's Surface

the digamma function ψ is defined as the logarithmic derivative of the gamma function:

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

We denote by $\varphi_y(x)$ the function

$$\varphi_y(x) = \psi(x+y+1) - \psi(x+1)$$

and calculate the coefficients of the first fundamental form, the unit normal vector to the *Pascal's Surface*, as well as the coefficients of the second fundamental form [3].

Thus, the mean curvature H and the Gauss curvature G have the following expressions:

$$H = \frac{g_{22}h_{11} - 2g_{12}h_{12} + g_{11}h_{22}}{2\det g} =$$

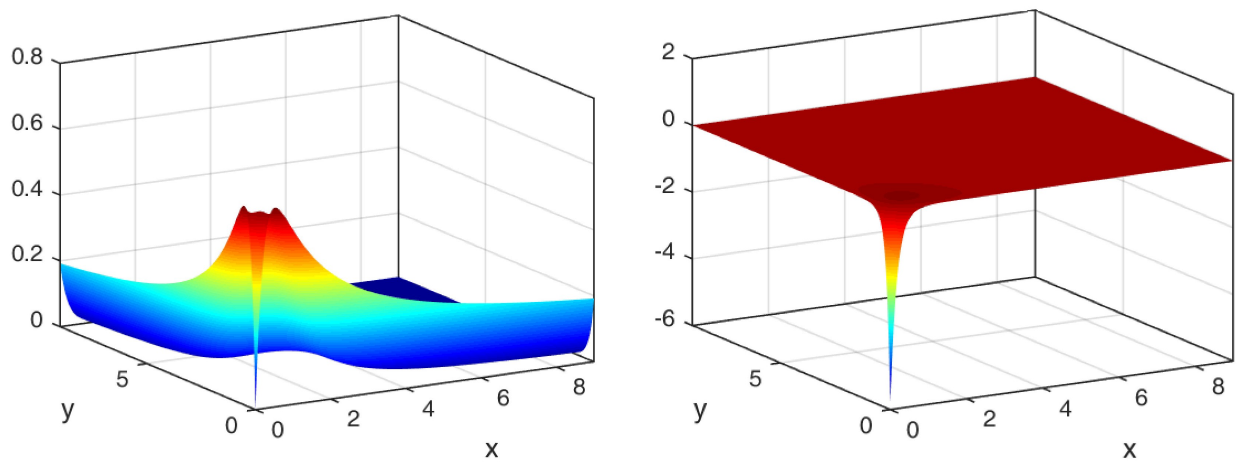
$$= \frac{z(x,y)}{2(\det g)^{3/2}} \left[\varphi_x^2(y) + \varphi_y^2(x) + \varphi_x'(y)\left(1 + z^2(x,y)\varphi_y^2(x)\right) + \varphi_y'(x)\left(1 + z^2(x,y)\varphi_x^2(y)\right) \right.$$
$$\left. -2z^2(x,y)\varphi_x(y)\varphi_y(x)\psi'(x+y+1)\right],$$

$$G = \frac{h_{11}h_{22} - h_{12}^2}{\det g} == \frac{z^2(x,y)}{\det^2 g}\left[\left(\varphi_y^2(x) + \varphi_y'(x)\right)\left(\varphi_x^2(y) + \varphi_x'(y)\right) - \left(\varphi_y(x)\varphi_x(y) + \psi'(x+y+1)\right)^2\right],$$

where

$$\det g = 1 + z^2(x,y)\left(\varphi_x^2(y) + \varphi_y^2(x)\right).$$

The graphs of the functions $H(x,y)$ and $G(x,y)$ are presented in Figure 4.



(**a**) (**b**)

Figure 4: (**a**) The mean curvature $H(x, y)$; (**b**) the Gauss curvature $G(x, y)$ of the *Pascal's Surface*

3. A Catalan triangle

The Catalan numbers are one of the most well-known sequences of positive integers, closely related to the binomial coefficients in the Pascal's triangle. Richard Stanley [9] collected 214 combinatorial interpretations of Catalan numbers, illustrating their ubiquity. The book contains also a history of the multiple (re)discoveries of Catalan numbers (written by Igor Pak). The most important combinatorial interpretations of Catalan numbers are synthesised by the following theorem:

Theorem 3.1 The Catalan number C_n counts the following:

(i) Triangulations of a convex polygon with $n + 2$ vertices.

(ii) Binary trees with n vertices.

(iii) Plane trees with $n + 1$ vertices.

(iv) Bracketings of a string of $n + 1$ identical characters x subject to a nonassociative binary operation.

(v) Ballot sequences of length $2n$.

(vi) Dyck paths of length $2n$.

The mathematical expression of these magnificent numbers is

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \tag{4}$$

or, equivalently,

$$C_n = \frac{(2n)!}{n!(n+1)!} = \binom{2n}{n} - \binom{2n}{n-1}. \tag{5}$$

The Catalan numbers are closely related to the *ballot numbers*. These numbers occur in the solution of the *ballot problem*, which can be formulated as follows: two candidates P and Q receive in an election p and q votes respectively; supposing that P wins $(p > q)$, what is the probability that P stays (strictly) ahead of Q during the counting of votes? The solution was given by J. Bertrand in

1887 and was also found by D. André using the *reflection principle* and reformulating the problem in terms of lattice paths: a counting of votes such that P stays (strictly) ahead of Q corresponds to a lattice path form $(0,0)$ to (p,q) with steps $(1,0)$ and $(0,1)$, staying under the line $y=x$ (and never touching it, except for $(0,0)$).

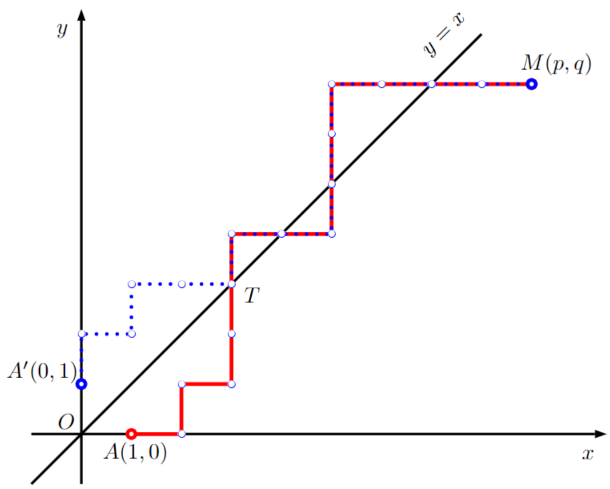


Figure 5: Reflection of the part AT with respect to the line $y=x$.

Consider (see Figure 5) a lattice path from A to M that touches (or crosses) the line $y=x$ at the point $T(k,k)$ for the first time. The part from A to T of the lattice path is reflected with respect to the line $y=x$. Thus, there exists a one-to-one correspondence between the lattice paths from A' to M and the lattice paths from A to M that have (at least) one vertex on the line $y=x$.

Since the number of lattice paths from $A'(0,1)$ to $M(p,q)$ is equal to $\binom{p+q-1}{p}$, it follows that the number of lattice paths that do not touch or cross the line $y=x$ is equal to

$$B(p,q)=\binom{p+q-1}{p-1}-\binom{p+q-1}{p}=\frac{p-q}{p+q}\binom{p+q}{p}. \tag{6}$$

By dividing the number of favourable cases (6) to the total number of possible cases, we obtain the required probability $\frac{p-q}{p+q}$.

The numbers $B(p,q)$ are known as *ballot numbers*.

A *ballot sequence* of length $2n$ is a sequence of n 1's and n -1's, such that every partial sum is nonnegative. From the relation above we obtain that the number of ballot sequences of length $2n$ is the Catalan number C_n:

$$B(n+1,n)=\frac{1}{n+1}\binom{2n}{n}=C_n.$$

If we write the numbers $B(p,q)$, for every $p=1,2,\ldots$ and $q=0,1,\ldots,p$ we obtain a triangle where the sequence of Catalan numbers appears twice (see ()).

$$\begin{array}{lllllllll}
\mathbf{1} & & & & & & & & \\
\mathbf{1} & \mathbf{1} & & & & & & & \\
1 & \mathbf{2} & \mathbf{2} & & & & & & \\
1 & 3 & \mathbf{5} & \mathbf{5} & & & & & \\
1 & 4 & 9 & \mathbf{14} & \mathbf{14} & & & & \\
1 & 5 & 14 & 28 & \mathbf{42} & \mathbf{42} & & & \\
1 & 6 & 20 & 48 & 90 & \mathbf{132} & \mathbf{132} & & \\
1 & 7 & 27 & 75 & 165 & 297 & \mathbf{429} & \mathbf{429} & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array} \tag{7}$$

This triangle is known as the *Catalan triangle*, being recorded as the sequence **A009766** in the On-line Encyclopedia of Integer Sequences [8]. As a matter of fact, there exist several triangles known as the *Catalan triangle*, but it seems that this is the most-standing form.

4. A Surface Associated to the Catalan Triangle

In the same way we extended the Pascal's triangle to obtain the Pascal's surface, we can extend the Catalan triangle to obtain the continuous surface (see Figure 6) defined in the 3-dimensional Euclidean space by the function

$$f_1 : \mathbb{R}_+^2 \setminus \{(0,0)\} \to \mathbb{E}^3, \quad f_1(x,y) = (x, y, z_1(x,y)),$$

where

$$z_1(x,y) = \frac{(x-y)\Gamma(x+y)}{\Gamma(x+1)\Gamma(y+1)} = \frac{x-y}{x+y} \cdot z(x,y). \tag{8}$$

It can be also written as

$$z_1(x,y) = \frac{\Gamma(x+y)}{\Gamma(x)\Gamma(y+1)} - \frac{\Gamma(x+y)}{\Gamma(x+1)\Gamma(y)} = z(x-1,y) - z(x,y-1).$$

If x and y are nonnegative integers, then $z_1(x,y) = \frac{x-y}{x+y}\binom{x+y}{x}$, which are exactly the numbers in the Catalan triangle (), completed (for $x \le y$) to an (infinite) antisymmetric matrix:

$$\begin{array}{cccccccccc}
* & \mathbf{-1} & \mathbf{-1} & -1 & -1 & -1 & -1 & -1 & -1 & \dots \\
\mathbf{1} & 0 & \mathbf{-1} & \mathbf{-2} & -3 & -4 & -5 & -6 & -7 & \dots \\
\mathbf{1} & \mathbf{1} & 0 & \mathbf{-2} & \mathbf{-5} & -9 & -14 & -20 & -27 & \dots \\
1 & \mathbf{2} & \mathbf{2} & 0 & \mathbf{-5} & \mathbf{-14} & -28 & -48 & -75 & \dots \\
1 & 3 & \mathbf{5} & \mathbf{5} & 0 & \mathbf{-14} & \mathbf{-42} & -90 & -165 & \dots \\
1 & 4 & 9 & \mathbf{14} & \mathbf{14} & 0 & \mathbf{-42} & \mathbf{-132} & -297 & \dots \\
1 & 5 & 14 & 28 & \mathbf{42} & \mathbf{42} & 0 & \mathbf{-132} & \mathbf{-429} & \dots \\
1 & 6 & 20 & 48 & 90 & \mathbf{132} & \mathbf{132} & 0 & \mathbf{-429} & \dots \\
1 & 7 & 27 & 75 & 165 & 297 & \mathbf{429} & \mathbf{429} & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array} \tag{9}$$

We remark that the function $z_1(x,y)$ cannot be extended by continuity at $(0,0)$:

$$\lim_{x\to 0} z_1(x,0) = 1, \text{ and } \lim_{y\to 0} z_1(0,y) = -1,$$

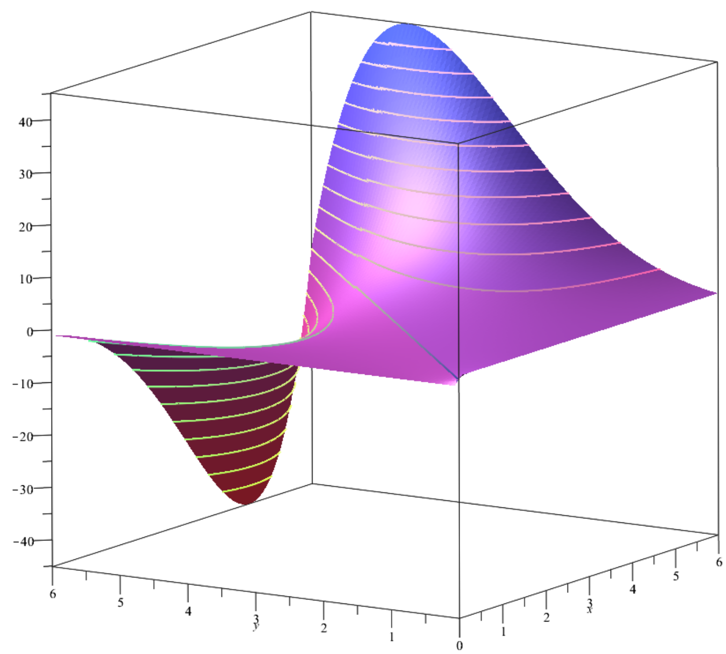


Figure 6: The surface associated to the Catalan triangle

so the limit $\lim_{(x,y)\to(0,0)} z_1(x,y)$ does not exist.

Since $z_1(y,x) = -z_1(x,y)$, for any x, y, we remark the symmetry of the surface w.r.t. the straight line

$$L_1: \ x = y, z = 0.$$

We also remark that for $x = y+1$ and $x = y+2$ respectively, where $y = n \in \mathbb{N}$, the Catalan numbers are obtained:

$$z_1(n+1,n) = C_n, \quad z_1(n+2,n) = C_{n+1}.$$

Similarly, for $y = x+1$ and $y = x+2$ respectively, ($x = n \in \mathbb{N}$) we obtain:

$$z_1(n,n+1) = -C_n, \quad z_1(n,n+2) = -C_{n+1}.$$

Besides the axis of symmetry, the surface contains four more straight lines:

$$L_2: y = 0, \ z = 1$$
$$L_3: y = 1, \ z = x - 1$$
$$L_4: x = 0, \ z = -1$$
$$L_5: x = 1, \ z = 1 - y$$

Moreover, one can prove that there are no more than these 5 lines entirely contained in the surface!

Theorem 4.1 The lines L_i, $i = 1, \ldots, 5$ are the only straight lines contained in the surface (S).

These geometrical properties of the surface associated to the Catalan triangle are illustrated in Figure 7.

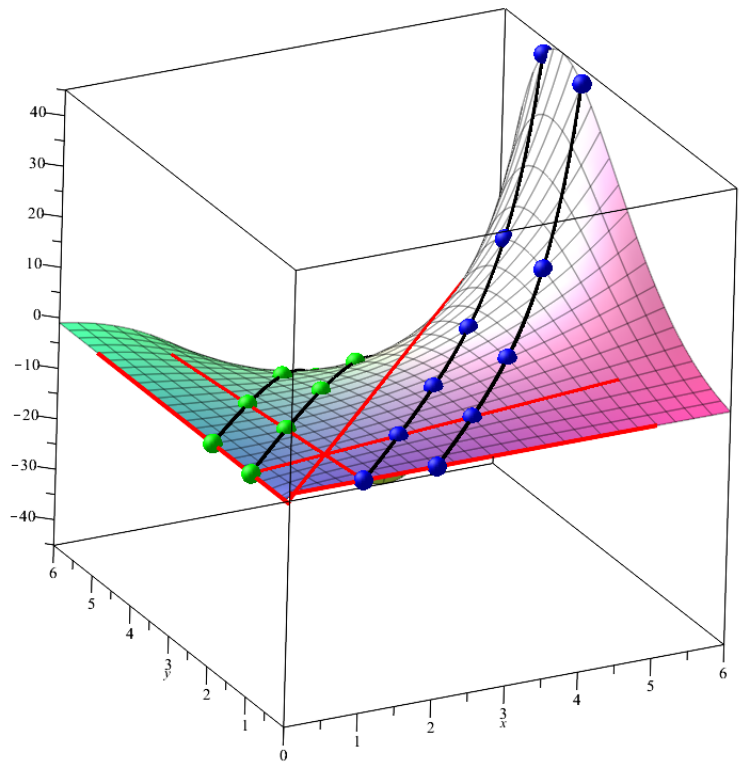


Figure 7: The points corresponding to Catalan numbers C_n (the blue ones) and negative Catalan numbers $-C_n$ (the green ones); the (red) lines $L_1, \ldots, L_5$ contained in the surface associated to the Catalan triangle.

More general, we have the following result regarding the cross-section of the surface with planes of the form $x = n \in \mathbb{N}$ and $y = n \in \mathbb{N}$, respectively.

Proposition 4.2 The curves of intersection of the surface related to Catalan triangle with planes of the form $x = n \in \mathbb{N}$ or $y = n \in \mathbb{N}$ are polynomials of degree n.

We denote by $\phi_y(x)$ the function

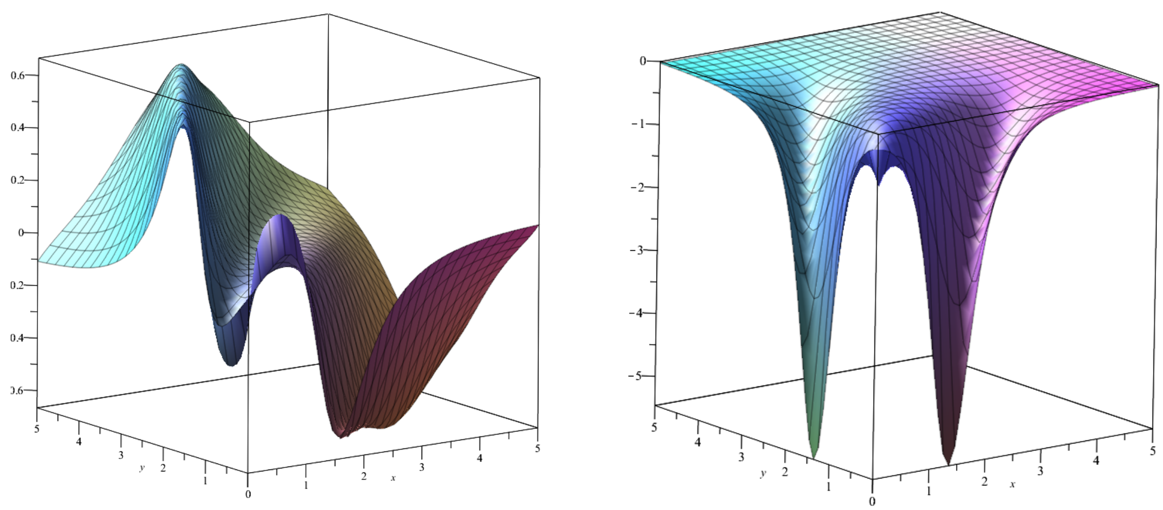
$$\phi_y(x) = \psi(x + y) - \psi(x + 1) + \frac{1}{x - y}.$$

By straightforward computations the expressions of the mean curvature H and the Gauss curvature G are obtained:

$$
\begin{aligned}
H &= \frac{g_{22}h_{11} - 2g_{12}h_{12} + g_{11}h_{22}}{2\det g} \\
&= \frac{z(x,y)}{2(\det g)^{3/2}} \left[\phi_x^2(y) + \phi_y^2(x) + \phi_x'(y)\left(1 + z^2(x,y)\phi_y^2(x)\right) + \phi_y'(x)\left(1 + z^2(x,y)\phi_x^2(y)\right)\right. \\
&\quad \left. -2z^2(x,y)\phi_x(y)\phi_y(x)\left(\psi'(x+y) + \frac{1}{(x-y)^2}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
G &= \frac{h_{11}h_{22} - h_{12}^2}{\det g} \\
&= \frac{z^2(x,y)}{\det^2 g}\left[\left(\phi_y^2(x) + \phi_y'(x)\right)\left(\phi_x^2(y) + \phi_x'(y)\right) - \left(\phi_y(x)\phi_x(y) + \psi'(x+y) + \frac{1}{(x-y)^2}\right)^2\right],
\end{aligned}
$$

where $\det g = 1 + z^2(x,y)\left(\phi_x^2(y) + \phi_y^2(x)\right)$.



(a) **(b)**

Figure 8: (**a**) The mean curvature $H(x,y)$; (**b**) the Gauss curvature $G(x,y)$ of the surface associated to the Catalan triangle.

Remark 4.3 If $z(x,y)$ is an anti-symmetric function, that is,

$$z(y,x) = -z(x,y),$$

then the surface explicitly defined by $z = z(x,y)$ has the mean curvature $H(x,y)$ with the same property,

$$H(y,x) = -H(x,y),$$

while the Gauss curvature $G(x,y)$ is a symmetric function:

$$G(y,x) = G(x,y).$$

These properties are satisfied by the surface $z_1(x, y)$, as it can be easily observed in Figure 8, which presents the mean curvature and the Gauss curvature of the surface associated to the Catalan triangle.

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GEOMETRIC STRUCTURES ON G_2 MANIFOLDS AND HARVEY LAWSON SUBMANIFOLDS

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ABSTRACT

During the past few decades, M-theory, a "theory of everything" has become a very active and exciting area of research as a leading candidate to unify the four fundamental forces of nature - electromagnetism, gravity, the weak and strong nuclear forces. In this paper we will discuss manifolds with special holonomy, spaces whose infinitesimal symmetries play an important role in M-theory compactifications. We will begin by providing brief introductions to G_2 manifolds and their Harvey-Lawson submanifolds, followed by a survey of recent research exploring the relations between symplectic, contact, and calibrated structures on such manifolds.

***K*eywords** Special holonomy, almost contact structure, symplectic structure, G_2 manifold, Calabi-Yau manifold.

1. Introduction

In 10-dimensional superstring theory, it is conjectured that the additional dimensions of spacetime (10=4+6) form a 6-dimensional Calabi-Yau manifold. This implies that, in this string theory, the universe is locally represented as $\mathbb{R}^{1,3} \times X$, where $\mathbb{R}^{1,3}$ is Minkowski space-time, and X is a 6-dimensional (real) Calabi-Yau manifold. Similarly, in M-theory (considered a theory of everything), the extra dimensions of spacetime (11=4+7) are postulated to constitute a 7-dimensional G_2 manifold. This means that the universe, within the framework of M-theory, is locally modeled on $\mathbb{R}^{1,3} \times M$, where $\mathbb{R}^{1,3}$ is Minkowski space-time, and M is a 7-dimensional (real) G_2 manifold.

In this paper, we begin by providing brief introductions to Calabi-Yau and G_2 manifolds and then we explore geometric structures on G_2 manifolds.

A 6-dimensional Riemannian manifold has an SU(3)-structure if the structure group of its frame bundle is reduced to the compact, special unitary group SU(3). SU(3) is a Lie group consisting of 3×3 complex matrices with determinant 1 and unitary transformations:

$$SU(3) = \{A \in \mathbb{C}^{3\times 3} : A^{\dagger}A = I,\ \det(A) = 1\},$$

where $A^{\dagger}$ represents the conjugate transpose of matrix A, I is the identity matrix, and $\det(A)$ is the determinant of A. The condition $A^{\dagger}A = I$ ensures unitarity, and $\det(A) = 1$ ensures that the matrices have determinant 1. A Calabi-Yau manifold is a Kähler manifold with a holonomy group contained within SU(n), where SU(n) is the special unitary group of complex n × n matrices. In this context, our focus is on the case where n=3 for the Calabi-Yau manifold.

Similarly, a 7-dimensional Riemannian manifold M has a G_2-structure if the structure group of its frame bundle is reduced to the compact, exceptional Lie group G_2. This condition implies that M is orientable and admits a spin structure. Alternatively, it means that the first and second Stiefel-Whitney classes of M vanish. A 7-dimensional Riemannian manifold (M, g) is called a G_2 manifold if the holonomy group of its Levi-Civita connection for the metric g is contained within $G_2 \subset SO(7)$.

A manifold with holonomy G_2 was first introduced by E. Bonan in 1966, showing that this manifold should be Ricci-flat [6]. In 1982, R. Harvey and B. Lawson studied the geometric structures on G_2 manifolds [13]. In 1989, R. Bryant and S. Salamon constructed the first examples of non-compact manifolds with G_2 holonomy [9]. Physicists are highly interested in G_2 manifolds due to their crucial role in M-theory compactifications.

More recently, in collaboration with F. Arikan and H. Cho [5], we showed that any 7-manifold with a spin structure, and thus a G_2-structure, also admits a compatible almost contact metric structure. Additionally, we showed that certain classes of G_2-manifolds have a contact structure. For further details on geometric structures of G_2 manifolds and their applications, refer to [5], [8], [7], [10], [11], [12], and [13].

An Open Problem:

Calabi-Yau Theorem: If M is a compact Kähler manifold with Kähler metric g and Kähler form ω , and R is any (1,1)-form representing the manifold's first Chern class, then there exists a unique Kähler metric $\tilde{g}$ on M with Kähler form $\tilde{\omega}$ such that ω and $\tilde{\omega}$ represent the same class in cohomology $H^2(M, \mathbb{R})$ and the Ricci form of $\tilde{\omega}$ is R.

S.T. Yau proved this theorem (The Calabi Conjecture) in 1978, through the solvability of the inhomogeneous complex Monge-Ampere equations on compact Kähler manifolds.

One important advance in differential geometry will be to state and prove a theorem analogous to the one for G_2 manifolds, which is currently an open problem. This will also contribute to our understanding of the topological obstructions to the existence of G_2 holonomy (Ricci flat) metrics on 7-manifolds.

2. G_2-structures

In this section we review the basics of G_2 geometry. More on the subject can be found in [1], [2], [3], [4], [8], and [13].

The set of octonions $\mathbb{O} = \mathbb{H} \bigoplus l\mathbb{H} = \mathbb{R}^8$ gives an 8 dimensional division algebra generated by $\{1, i, j, k, l, li, lj, lk\}$. The set of imaginary octonions $im(\mathbb{O}) = \mathbb{R}^7$ has a cross product operation $\times : \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$, defined by $u \times v = \mathrm{Im}(v \cdot u)$, where $\cdot$ is the octonionic multiplication. The exceptional Lie group G_2 is the linear automorphisms of the imaginary octonions $im(\mathbb{O}) \cong \mathbb{R}^7$ preserving this cross product.

By a theorem of Schouten, [8], [17], the group G_2 can also be defined as the subgroup of $GL(7, \mathbb{R})$ which preserves the 3-form $\varphi_0 \in \Omega^3(\mathbb{R}^7)$,

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},$$

where $(x_1, ..., x_7)$ are the coordinates on $\mathbb{R}^7$ and $e^{ijk} = dx_i \wedge dx_j \wedge dx_k$. That is,

$$G_2 = \{A \in GL(7, \mathbb{R}) | A^* \varphi_0 = \varphi_0\}.$$

Then we can define the manifold with G_2 structure:

Definition 2.1 A *manifold with G_2 structure* is a smooth 7-dimensional manifold M such that the structure group of M reduces to the exceptional Lie group G_2. Equivalently, it is a 7-dimensional manifold M that admits a nondegenerate 3-form $\varphi \in \Omega^3(M)$ such that at any point $p \in M$,

$$(T_pM, \varphi_p) \cong (\mathbb{R}^7, \varphi_0).$$

A manifold with G_2 structure determines a metric g and a cross product $\times$ on M such that

$$\varphi(u, v, w) = g(u \times v, w), \;\; u, v, w \in TM.$$

Definition 2.2 Suppose (M, φ) is a manifold with G_2 structure. We call (M, φ) a *G_2-manifold* if φ is covariantly constant with respect to the Levi-Civita connection.

Note that the covariantly constant condition is equivalent to saying that the form φ is closed and co-closed, i.e.

$$d\varphi = d \star \varphi = 0.$$

In 1982, Harvey and Lawson called φ and $\star\varphi$ the calibration 3-form and 4-form, respectively, when they introduced *calibrated submanifolds*. They showed that calibrated submanifolds are volume-minimizing in their homology class.

Definition 2.3 A *calibration* is a closed p-form ϕ on a Riemannian manifold X^n such that ϕ restricts to each oriented tangent p-plane of X^n to be less than or equal to the volume form of that p-plane.

Definition 2.4 The submanifolds of X^n for which the p-form ϕ restricts to be equal to the Riemannian volume form are called to be *calibrated* by the form ϕ.

The term *calibrated geometry* represents the ambient manifold X, the calibration ϕ, and the collection of submanifolds calibrated by ϕ.
There are two types of calibrated submanifolds of G_2 manifolds.
Definition 2.5 Let (M, φ) be a G_2 manifold with calibration 3-form φ. A 4-dimensional submanifold $C \subset M$ is called *coassociative* if $\varphi|_C = 0$. A 3-dimensional submanifold $A \subset M$ is called *associative* if $\varphi|_A = dvol(A)$.

Note that the condition $\varphi|_A = dvol(A)$ is equivalent to the condition that $\chi|_A \equiv 0$, where $\chi \in \Omega^3(M, TM)$ is the tangent bundle-valued 3-form defined by the identity:

$$\langle \chi(u, v, w), z \rangle = *\varphi(u, v, w, z).$$

The equivalence of these conditions follows from the 'associator equality' of

$$\varphi(u, v, w)^2 + |\chi(u, v, w)|^2/4 = |u \wedge v \wedge w|^2.$$

Then one can define "Harvey-Lawson" submanifolds, [4]:
Definition 2.6 A *Harvey-Lawson manifold* is a 3-dimensional submanifold $HL \subset M$ of a G_2 manifold such that

$$\varphi|_{HL} = 0$$

Equivalently, this is defined by $\langle \chi|_{HL}, \chi|_{HL} \rangle = 1$.

One can also define a tangent bundle-valued 2-form, which is just the cross product of M, [3].

Definition 2.7 Let (M, φ) be a G_2 manifold. Then $\psi \in \Omega^2(M, TM)$ is the tangent bundle-valued 2-form defined by the identity

$$\langle \psi(u, v), w \rangle = \varphi(u, v, w) = \langle u \times v, w \rangle.$$

Note that at any point on M, there exists an orthonormal frame such that tangent bundle-valued forms χ and ψ can be expressed as follows:

$$\begin{aligned}
\chi = \; & (e^{256} + e^{247} + e^{346} - e^{357})e_1 \\
& + (-e^{156} - e^{147} - e^{345} - e^{367})e_2 \\
& + (e^{157} - e^{146} + e^{245} + e^{267})e_3 \\
& + (e^{127} + e^{136} - e^{235} - e^{567})e_4 \\
& + (e^{126} - e^{137} + e^{234} + e^{467})e_5 \\
& + (-e^{125} - e^{134} - e^{237} - e^{457})e_6 \\
& + (-e^{124} + e^{135} + e^{236} + e^{456})e_7,
\end{aligned}$$

and

$$\begin{aligned}\psi = \ & (e^{23} + e^{45} + e^{67})e_1 \\ & + (e^{46} - e^{57} - e^{13})e_2 \\ & + (e^{12} - e^{47} - e^{56})e_3 \\ & + (e^{37} - e^{15} - e^{26})e_4 \\ & + (e^{14} + e^{27} + e^{36})e_5 \\ & + (e^{24} - e^{17} - e^{35})e_6 \\ & + (e^{16} - e^{25} - e^{34})e_7\end{aligned}$$

where $e_1, ..., e_7$ is the local orthonormal frame with dual frame $e^1, ..., e^7$.

3. Harvey-Lawson Submanifolds

It is well-known that a smooth symplectic manifold N^{2n} admits a closed, non-degenerate differential 2-form ω. An n-dimensional submanifold L^n of N^{2n} is called Lagrangian if the restriction of ω to L is zero. In symplectic geometry, there is a famous quote by Alan Weinstein which says, 'Everything is Lagrangian,' because every manifold is the Lagrangian zero section of its cotangent bundle. In our recent paper [15], as a new quote, we stated that 'Every closed, oriented, smooth 3-manifold is Harvey-Lawson,' and proved the following:

Theorem 3.1 Let (Y^3, g) be a closed, oriented, real analytic Riemannian 3–manifold. Then there exists a G_2–manifold M^7 and an isometric embedding $i : Y \hookrightarrow M$ such that the image $i(Y)$ is a HL–submanifold, contained in a compact coassociative submanifold of M.

In [8], R. Bryant proved that every closed, oriented, real analytic Riemannian 4-manifold whose bundle of self-dual 2-forms is trivial can be isometrically embedded as a coassociative submanifold in a G_2-manifold, even as the fixed locus of an anti-G_2 involution. Moreover, there exists a tubular neighborhood of $i(Y)$ in M which is trivial.

Let (Y^3, g) be a closed, oriented, real analytic Riemannian 3–manifold. Then the oriented 4-manifold $Y \times S^1$ is spin and has zero signature. Moreover the Euler characteristic is also zero. These imply that the bundle of self-dual 2-forms on $Y \times S^1$ is topologically trivial. By [8], $Y \times S^1$ can be isometrically embedded as a coassociative submanifold in a G_2-manifold. Therefore, Y will be a Harvey-Lawson submanifold of this G_2-manifold.

We now study the normal bundle of a HL submanifold inside a G_2 manifold M. An orthonormal 3-frame field $\Gamma = \{u, v, w\}$ on (M, φ) is called a G_2-*frame field* if $\varphi(u, v, w) = \langle u \times v, w\rangle = 0$, [1], [2]. It is also well known that there always exists a nonvanishing 2-frame field $\Lambda = \{u, v\}$ on a manifold with G_2 structure, [18].

Let $TM = \mathbb{E} \oplus \mathbb{V}$ be the splitting with $\mathbb{E} = \{u, v, u \times v\}$ and the corresponding $\mathbb{V}$. Let w be a unit section of the bundle $\mathbb{V} \to M$. The existence of w on the entire manifold M is not guaranteed, but one can show that w exists on a tubular neighborhood of the 3-skeleton $M^{(3)}$ of M which is the complement of a 3 complex $Y \subset M$. Let $\Gamma = \{u, v, w\}$ be an orthonormal G_2 frame field satisfying $\varphi(u, v, w) = \langle u \times v, w \rangle = 0$, and we consider another non-vanishing vector field defined as

$$R = \chi(u, v, w) = -u \times (v \times w).$$

One can show that the following properties hold ([4]):

(a) If $\{u, v, w\}$ is a HL 3-plane field, then $\mathbb{V} = \{u, v, w, R\}$ is a coassociative 4-plane field.
(b) $\mathbb{E} = \{u \times v, v \times w, w \times u\}$ is an associative 3-plane field.
(c) $\mathbb{E} \perp \mathbb{V}$ and $\{u, v, w, R, u \times v, v \times w, w \times u\}$ is an orthonormal frame field on M.

Using the identification of vectors and 1-forms through the metric g, the 3-form φ can be expressed as

$$\begin{aligned}\varphi &= u \wedge v \wedge (u \times v) + v \wedge w \wedge (v \times w) + w \wedge u \wedge (w \times u) \\ &\quad + u \wedge R \wedge (v \times w) + v \wedge R \wedge (w \times u) + w \wedge R \wedge (u \times v) \\ &\quad - (u \times v) \wedge (v \times w) \wedge (w \times u).\end{aligned}$$

Moreover, since $\{u, v, w, R, u \times v, v \times w, w \times u\}$ and $\{u, v, w, R, w \times R, u \times R, v \times R\}$ are equivalent frames, φ can be written as

$$\begin{aligned}\varphi &= u \wedge v \wedge (w \times R) + v \wedge w \wedge (u \times R) + w \wedge u \wedge (v \times R) \\ &\quad + u \wedge R \wedge (u \times R) + v \wedge R \wedge (v \times R) + w \wedge R \wedge (w \times R) \\ &\quad - (w \times R) \wedge (u \times R) \wedge (v \times R).\end{aligned}$$

Using this expression for φ, the normal bundle of an HL submanifold can then be decomposed as

$$N(HL) = \tilde{N}(HL) \oplus R,$$

where $\tilde{N}(HL)$ is generated by vector fields $u \times R$, $v \times R$, and $w \times R$. Since $\langle \psi(u, v), w \rangle = \varphi(u, v, w) = \langle u \times v, w \rangle = 0$ for an HL submanifold, $\tilde{N}(HL)$ is isomorphic to $T(HL)$. The cross product structure $\times$ (also known as ψ) induces this isomorphism. Note that every oriented 3 manifold is parallelizable. So $T(HL)$ is trivial. And hence $\tilde{N}(HL)$ is trivial. So $N(HL) = \tilde{N}(HL) \oplus \mathbb{R}$ is trivial.

Next, we study the deformations of HL-submanifolds in a G_2 manifold. For more on the subject, see [12]. Recall that McLean studied deformations of compact special Lagrangian submanifolds in Calabi-Yau manifolds, [14]. Using the same approach we can prove the following theorem [12].

Theorem 3.2 The space of infinitesimal deformations of a smooth, compact, orientable 3-dimensional Harvey-Lawson submanifold HL in a G_2 manifold M within the class of HL-submanifolds is infinite-dimensional. This space can be identified with the direct sum of the spaces of smooth functions and closed 2-forms on HL.

Proof. For a small vector field V, the deformation map is a map, F, defined from an open neighborhood of zero in the space of sections of the normal bundle, $U \subset \Gamma(N(HL))$, to the space of differential 3-forms on HL, $\Lambda^3 T^*(HL)$, such that

$$F : U \to \Lambda^3 T^*(HL),$$
$$F(V) = (\exp_V)^*(\varphi|_{HL_V}).$$

This means that the deformation map F first restricts φ to HL_V and then pulls this back to HL via $(\exp_V)^*$ where $\exp_V$ is the normal exponential map that gives a diffeomorphism of HL onto its image HL_V in a neighborhood of 0.

Since HL is compact, normal vector fields can be identified with nearby submanifolds. Under this identification, the kernel of F then corresponds to HL deformations.

The linearization of F at 0 is given by

$$dF(0) : \Gamma(N(HL)) \to \Lambda^3 T^*(HL)$$

where

$$\begin{aligned} dF(0)(V) =& \frac{\partial}{\partial t} F(tV)|_{t=0} = \frac{\partial}{\partial t}[\exp^*_{tV}(\varphi)] \\ =& \left[\mathcal{L}_V(\varphi)|_{HL}\right]. \end{aligned}$$

Further, by Cartan's formula, we have

$$\begin{aligned} dF(0)(V) &= (\iota_V d\varphi + d(\iota_V \varphi))|_{HL} \\ &= d(\iota_V \varphi)|_{HL}, \end{aligned}$$

where ι_V represents the interior derivative. Using the decomposition of the normal bundle, $N(HL) = \tilde{N}(HL) \oplus \langle R \rangle$, we write

$$V = V' + V_R R$$

Note that V' is isomorphic to a vector field $V'' \in T(HL)$ since $\tilde{N}(HL) \cong T(HL)$. Therefore, we have

$$\begin{aligned} dF(0)(V) &= d((\iota_{V'}\varphi) + V_R(\iota_R \varphi))|_{HL} \\ &= d(\iota_{V'}\varphi)|_{HL} \\ &= d(\star\eta), \end{aligned}$$

where η is the dual 1-form to the vector field V'' with respect to the induced metric, and $\star\eta$ is the Hodge dual of η on HL. Hence

$$dF(0)(V) = d(\star\eta) = d^*\eta.$$

Therefore, the set of nontrivial deformations for HL-submanifolds can be identified with closed 2-forms on HL. However, this description omits additional deformations—specifically, the trivial deformations arising from the R component of the vector field. These correspond to the deformation of a 3-dimensional HL manifold within a coassociative submanifold. By definition, any such 3-manifold will be HL, implying that deformations of HL within a coassociative submanifold in the direction of R can be identified with smooth functions on HL.

4. Contact and Almost Contact Manifolds

In this section we will review the properties of (almost) contact structures on manifolds with G_2 structure and show that every manifold with G_2 structure is almost contact. For more on the subject see [5].

Let M be a $(2n+1)$-dimensional smooth manifold. A plane field (or hyperplane distribution) ξ on M can (locally) be given as the kernel of 1-form $\alpha : \xi_x = \ker(\alpha_x)$, $x \in M$.

Definition 4.1 A contact structure on M is a hyperplane field ξ that is (locally) given by the kernel of a 1-form α such that $\alpha \wedge (d\alpha)^n \neq 0$. The pair (M, ξ) is called a contact manifold.

Definition 4.2 An almost contact structure on an odd-dimensional differentiable manifold (M^{2n+1}, J, R, α) consists of a field J of endomorphisms of the tangent spaces, a vector field R, and a 1-form α satisfying the following conditions:

$$\text{(i) } \alpha(R) = 1,$$

$$\text{(ii) } J^2 = \text{- id} + \alpha \otimes R.$$

Here id denotes the identity transformation.

Definition 4.3 An almost contact metric structure on an odd-dimensional differentiable manifold $(M^{2n+1}, J, R, \alpha, g)$ consists of an almost contact structure (J, R, α) and a Riemannian metric g satisfying

$$g(Ju, Jv) = g(u, v) - \alpha(u)\alpha(v)$$

for all vector fields u, v in TM. In this case such a g is called a compatible metric.

Theorem 4.1 (A-C-S) Let (M^7, φ) be a manifold with G_2 structure. Then M admits an almost contact structure. Moreover, for any non-vanishing vector field R on M, $(J, R, \alpha_R, \langle \cdot, \cdot \rangle_\varphi = g_\varphi)$ is an almost contact metric structure on M, [5].

Proof. Here we give an explicit construction of the almost contact structure. More can be found in [5]. Let (M, φ) be a manifold with G_2-structure. As M is 7-dimensional there exists a nowhere vanishing vector field R on M. Let $\langle \cdot, \cdot \rangle_\phi$ denotes the Riemannian metric and $\times_\varphi$ denotes the cross product determined by φ. Using the metric, we define the 1-form α as the metric dual of R, that is,

$$\alpha(u) = \langle R, u \rangle_\varphi.$$

The cross product $\times_\varphi$ and R defines an endomorphism $J_R : TM \to TM$ of the tangent spaces by

$$J_R(u) = R \times_\varphi u.$$

We take the structure (J_R, R, α_R) described as above. Then we get

$$\alpha_R(R) = g_\varphi(R, R) = 1$$

and

$$J_R^2(u) = J_R(R \times_\varphi u) = R \times_\varphi (R \times_\varphi u) = -|R|^2 u + g_\varphi(R, u)R.$$

So we conclude that (J_R, R, α_R) is an almost contact structure on (M, φ).

Note that $J_R(R) = 0$, and so J_R, indeed, defines a complex structure on the orthogonal complement $R^\perp$ of R with respect to $\langle \cdot, \cdot \rangle_\varphi$.

Next, we show that g_φ=$\langle \cdot, \cdot \rangle_\varphi$ is a compatible metric with an almost contact structure (J_R, R, α_R). In order to show this, we compute $g(Ju, Jv)$ and show that it is equal to $g(u, v) - \alpha(u)\alpha(v)$ for all $u, v \in TM$.

$$\begin{aligned}
\langle J_R u, J_R v\rangle_\varphi &= \langle R \times u, R \times v\rangle = \varphi(R, u, R \times v) \\
&= -\varphi(R, R \times v, u) = -\langle R \times (R \times v), u\rangle \\
&= -\langle -|R|^2 v + \langle R, v\rangle R, u\rangle = \langle |R|^2 v, u\rangle - \langle \langle R, v\rangle R, u\rangle\rangle \\
&= \langle |R|^2 v, u\rangle - \langle \alpha(v) R, u\rangle = \langle u, |R|^2 v\rangle - \alpha(v)\langle R, u\rangle \\
&= \langle u, |R|^2 v\rangle - \alpha(v)\alpha(u)
\end{aligned}$$

And hence $g(Ju, Jv) = g(u, v) - \alpha(u)\alpha(v)$ for all $u, v \in TM$.

Note that in the special case where $u = R$ (or $v = R$) then

$\langle J_R R, J_R v\rangle = \langle 0, J_R v\rangle = 0$ and $\langle R, v\rangle - \alpha(R)\alpha(v) = \alpha(v) - \alpha(v) = 0$

And if u, v are both taken from the orthogonal complement $R^\perp$ (wrt $\langle \cdot, \cdot \rangle_\varphi$), then

$$\begin{aligned}
\langle J_R u, J_R v\rangle &= \langle R \times u, R \times v\rangle = \varphi(R, u, R \times v) = -\varphi(R, R \times v, u) \\
&= -\langle R \times (R \times v), u\rangle = -\langle -|R|^2 v + \langle R, v\rangle R, u\rangle \\
&= -\langle -v, u\rangle = \langle u, v\rangle
\end{aligned}$$

We now prove an extension theorem that states the relations between contact structures on submanifolds (associative and Harvey-Lawson) and the ambient G_2 manifold.

Theorem 4.2 There is an almost contact structure on a G_2-manifold M coming from an (almost) contact structure on an associative 3-manifold Y. This also holds for 3-dimensional HL submanifolds of M.

Proof. Let Y be an associative (or HL) 3-dimensional submanifold of M. Since Y is 3-dimensional there always exists a contact structure ξ_Y(= ker α_Y) and a corresponding almost contact metric structure $(J_Y, R_Y, \alpha_Y, g_Y)$. Here J_Y is an almost complex structure on ξ_Y.

Note that the Stiefel manifold $V_k(\mathbb{R}^n)$, the set of all orthonormal k-frames in $\mathbb{R}^n$ is equivalent to the set of ordered orthonormal k-tuples of vectors in $\mathbb{R}^n$. It is an $n - k - 1$ connected, compact manifold whose dimension is given by $nk - 1/2k(k+1)$, (4-connected for $n = 7$ and $k = 2$).

One can use classical obstruction theory to show that there is no obstruction for the homotopy extension theorem. Here we have the fibers given by $V_2(\mathbb{R}^7)$. If M has a 2-frame field, then this implies that the fiber bundle $V_2(\mathbb{R}^7)$ has a section. One can check when two such sections of this fiber bundle are homotopic to each other using obstruction theory. The number of the sections of $V_2(\mathbb{R}^7)$ is given by $H^i(Y, \pi_i(F))$, where F is the fiber. Since Y is 3 dimensional and $V_2(\mathbb{R}^7)$ is 4-connected these groups are zero up to homotopy, in another words, one can always deform one section to the other.

Now, let ξ_Y be the 2-plane spanned by two nonzero vectors u, v. By homotopy extension theorem we can extend them to the ambient manifold G_2 manifold M. Denote u' and v' by the extensions of u and v respectively. One can easily check that this extension of 2-plane field provides and almost contact structure on G_2 manifold.

Let's define a vector in M as $R' = u' \times v'$ and a linear transformation $J' : T_pM \to T_pM$ as $J'(u') = R' \times u'$. Then using the properties of the cross product we have

$$\begin{aligned} J'(u') &= (u' \times v') \times u' = -u' \times (u' \times v') \\ &= -(-|u'|^2 v' + \langle u', v' \rangle u') = v', \end{aligned}$$

and if we apply J' again

$$\begin{aligned} J'(J'(u')) &= J'(v') = R' \times v' = (u' \times v') \times v' \\ &= -v' \times (u' \times v') \\ &= v' \times (v' \times u') = -|v'|^2 u' + \langle v', u' \rangle v' = -u'. \end{aligned}$$

In general, one can show that for any nonzero vector w in the orthogonal complement of $R' \in T_pM$,

$$\begin{aligned} J'^2(w) &= J'(J'(w)) = J'((u' \times v') \times w) \\ &= (u' \times v') \times ((u' \times v') \times w) \\ &= -|u' \times v'|^2 w + \langle u' \times v', w \rangle (u' \times v') = -w. \end{aligned}$$

This implies that (J', R', α') is an almost contact structure of M for 1-form α' that satisfies $\langle R', \cdot \rangle_\varphi = g_\varphi(R', \cdot) = \alpha'(\cdot)$.

5. Interesting Questions

In this paper, we explored the relationships among G_2 structures, vector fields, and (almost) contact structures on G_2 manifolds and their Harvey-Lawson submanifolds. An intriguing avenue for further investigation lies in understanding the connections between Harvey-Lawson manifolds and the construction of certain 'mirror dual' Calabi-Yau submanifolds within a G_2 manifold. Specifically, when given a Harvey-Lawson manifold HL, it would be interesting to investigate how one can assign a pair of tangent bundle-valued 2 and 3-forms to a G_2 manifold $(M, HL, \varphi, \Lambda)$, where φ is the calibration 3-form and Λ is an oriented 2-plane field. As shown in [3], these forms can then be utilized to define different complex and symplectic structures on certain 6-dimensional subbundles of $T(M)$. Upon integration of these bundles, mirror Calabi-Yau manifolds are obtained. Additionally, an intriguing question is how contact and symplectic structures in Calabi-Yau and G_2 manifolds are related, along with the role of Harvey-Lawson submanifolds. Furthermore, it is very interesting to understand how these contact and symplectic structures will help to state and prove a theorem analogous to the one for G_2 manifolds—a theorem to find the topological condition to guarantee the existence of integrable G_2 structures, which is currently an 'open problem,' as mentioned earlier. We expect that these geometric structures will contribute to our understanding of the topological obstructions to the existence of G_2 holonomy (Ricci flat) metrics on 7-manifolds. In a future project, we plan to address these questions and extend the constructions to Spin(7) manifolds and their submanifolds, [16].

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HIGHER-POWER HARMONICITY

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ABSTRACT

We give a brief overview of the variational theory of the family of higher-power energy functionals for mappings between Riemannian manifolds, and its generalisation to sections of Riemannian fibre bundles, with particular emphasis on Riemannian vector bundles and their sphere bundles. The main example is Ramachandran's complete classification of all higher-power harmonic left-invariant vector fields on 3-dimensional unimodular Lie groups equipped with an arbitrary left-invariant Riemannian metric, which rounds out the picture for harmonic vector fields in these cases obtained by Gonzalez-Davila and Vanhecke [3].

***K*eywords** Riemannian manifolds, Elementary invariants, Newton polynomials, Higher-power energy and vertical energy, Newton tensors, 3-dimensional unimodular Lie groups

1. Introduction

The ideas of "higher-power energy" and "higher-power harmonic maps" appeared briefly in the foundational paper [1] of Eells and Sampson. However, despite the subsequent popularity of harmonic mappings, they made comparitively little headway, at least within the differential geometric community[74]. During my graduate studies (with Eells) I spent some time thinking about "higher-power harmonicity", and wrote up what I learned as the first part of my PhD thesis [10]. However, this material was not published. More recently, my graduate student Anand Ramachandran was able to develop some new angles on the topic in his PhD thesis [8]. We then decided to write a joint paper [9] combining results from both our theses, intended to be a foundation for the theory of "higher-power harmonic sections". Unfortunately Anand passed away before the paper was completed, so it now also stands as a memorial to him and his work.

This talk is a brief summary of [9], together with a few results that we were unable to fit in to that paper (Theorems , , and). For completeness, the proofs of these

[74] Mathematical physicists, independently, used of one of the higher-power energies to formulate the theory of "skyrmions".

additional results have been included here. (There are further results from [8] concerning non-unimodular 3-dimensional Lie groups that do not appear either here or in [9], and still await publication.) The intention is to give some flavour of the topic, without getting bogged down with too much technical detail.

2. Higher-power harmonic maps

We begin with some brief algebraic preliminaries. Suppose V is an m-dimensional (real) vector space, and α is a linear endomorphism of V. Writing the characteristic polynomial of α as follows:

$$\chi_\alpha(\lambda) = \varepsilon_m(\alpha) - \varepsilon_{m-1}(\alpha)\lambda + \cdots + (-1)^{m-1}\varepsilon_1(\alpha)\lambda^{m-1} + (-1)^m\varepsilon_0(\alpha)\lambda^m,$$

we have that:

$$\varepsilon_m(\alpha) = \det(\alpha),\ \ldots,\ \varepsilon_1(\alpha) = trace(\alpha),\ \varepsilon_0(\alpha) = 1.$$

In general, $\varepsilon_r(\alpha)$ is the *r-th elementary invariant* of α; if α is diagonalisable then $\varepsilon_r(\alpha)$ is simply the r-th elementary symmetric polynomial in the eigenvalues of α. In general $\varepsilon_r(\alpha)$ satisfies *the Newton-Girard identity*:

$$r\varepsilon_r(\alpha) = \varepsilon_{r-1}(\alpha)trace(\alpha) - \cdots + (-1)^{r-2}\varepsilon_1(\alpha)trace(\alpha^{r-1}) + (-1)^{r-1}trace(\alpha^r). \tag{1}$$

The *r-th Newton polynomial* of α ($r = 0, 1, \ldots, m$) is defined:

$$\varepsilon_{\alpha,r}(\lambda) = \varepsilon_r(\alpha) - \varepsilon_{r-1}(\alpha)\lambda + \cdots + (-1)^{r-1}\varepsilon_1(\alpha)\lambda^{r-1} + (-1)^r\lambda^r, \tag{2}$$

and the *r-th Newton tensor* of α is then the following linear map:

$$\nu_r(\alpha)\colon V \to V; \nu_r(\alpha) = \varepsilon_{\alpha,r}(\alpha).$$

We note that $\nu_0(\alpha) = I$ (the identity map), and $\nu_m(\alpha) = 0$ (the zero map) by the Cayley-Hamilton theorem.

Now suppose that $\varphi\colon (M, g) \to (N, h)$ is a smooth mapping of Riemannian manifolds, with $\dim(M) = m$ and $\dim(N) = n$. We apply the above algebraic constructs to the first fundamental tensor α of ϕ (also known as the *Cauchy-Green tensor*), defined:

$$g(\alpha(X), Y) = \varphi^*h(X, Y) = h(d\varphi(X), d\varphi(Y)), \tag{3}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. Then:

$$\varepsilon_1(\alpha) = \|d\varphi\|^2 = 2e(\varphi),\ \ldots,\ \varepsilon_m(\alpha) = v(\varphi)^2,$$

where $e(\varphi)$ is the energy density of φ, and $v(\varphi)$ is the volume density. In general we have:

$$\varepsilon_r(\alpha) = \|(d\varepsilon)^r\|^2 = \|d\varphi \wedge \cdots \wedge d\varphi\|^2.$$

Relabelling the elementary invariants as $\varepsilon_r(\varphi)$, it follows that $\varepsilon_r(\varphi)\colon M \to \mathbb{R}$ is a smooth function that at each point measures the average deformation by φ of infinitesimal r-dimensional volume (squared). If M is compact and orientable then the total amount of r-dimensional deformation imposed by φ is therefore:

$$\mathcal{E}_r(\varphi) = \frac{1}{2}\int_M \varepsilon_r(\varphi)vol(g), \quad r = 1, \ldots, m.$$

We refer to this as the *r-power energy*, or simply *r-energy*, of φ. Note that $\mathcal{E}_r(\varphi) = 0$ if and only if the rank of $d\varphi$ drops below r everywhere on M; in particular, the r-power energy is trivial if $r > n$.

Remark 2.1 [9, Proposition 2.5]. If $m = 2r$ then $\mathcal{E}_r(\varphi)$ depends only on the conformal class of g.

In order to decide which mappings are optimal with respect to r-power energy we use a variational approach, just as for the "classic" case $r = 1$. After applying the calculus of variations, this produces a non-linear system of second order partial differential equations characterising the critical points of the functional $\mathcal{E}_r \colon \mathfrak{C}^\infty(M, N) \to \mathbb{R}$ (see Theorem below). It turns out that these Euler-Lagrange equations can be written rather succinctly using the Newton tensors $\nu_r(\alpha)$, which we relabel as $\nu_r(\varphi)$ and refer to as the *Newton tensors of* φ. Since α is symmetric, the Newton tensors are symmetric $(1,1)$-tensors on M. We now define the *r-power tension field* of φ by:

$$\tau_r(\varphi) = trace\nabla(d\varphi \circ \nu_{r-1}(\varphi)).$$

When $r = 1$ we recover the standard tension field $\tau(\varphi)$ since $\nu_0(\varphi) = I$, and in all cases $\tau_r(\varphi)$ is a section of the pullback bundle $\varphi^{-1}TN \to M$.

It is possible to split the higher-power tension fields into two pieces as follows:

$$\tau_r(\varphi) = trace_\nu \nabla d\varphi + d\varphi(div\nu), \tag{4}$$

where for notational clarity we have abbreviated $\nu = \nu_{r-1}(\varphi)$, and $trace_\nu$ denotes the following "twisted trace":

$$trace_\nu \nabla d\varphi = \sum_{i=1}^{m} \nabla d\varphi(\nu E_i, E_i) = \sum_{i=1}^{m} d\varphi(E_i, \nu E_i), \tag{5}$$

for any local orthonormal tangent frame field $\{E_i\}$ in M. (The symmetry of ν ensures that its placement is unambiguous.) Since the coefficients of ν are homogeneous polynomials of degree $2r - 2$ in the first order partial derivatives of φ, when expressed in local coordinates, this clarifies the non-linearity of $\tau_r(\varphi)$ as a differential operator (without writing it out in full detail). Furthermore, if the Newton tensors of φ are solenoidal (ie. $div\nu = 0$) then $\tau_r(\varphi)$ simplifies dramatically. Unfortunately (or otherwise) this is not generally the case, and situations where it occurs are of great interest.

Theorem 2.2 [9, Theorem 2.12]. For any smooth variation φ_t of φ we have:

$$\frac{d}{dt}\Big|_{t=0} \mathcal{E}_r(\varphi_t) = -\int_M h\Big(\tau_r(\varphi), \frac{\partial \varphi_t}{\partial t}\Big|_{t=0}\Big) vol(g).$$

Thus φ is a critical point of $\mathcal{E}_r$ if and only if $\tau_r(\varphi) = 0$.

Theorem leads to the following definition.

Definition 2.3 A smooth mapping $\varphi \colon (M, g) \to (N, h)$ is a *r-power harmonic map*, or simply *r-harmonic map*, if $\tau_r(\varphi) = 0$, for $r = 1, \ldots, m$.

A full frontal attack on the r-harmonic map equations is certainly a daunting prospect. Instead, we look at specific geometric situations in which the equations become more manageable, perhaps the simplest being when either M or N is a sphere:

$$\mathbb{S}^k = \{x \in \mathbb{R}^{k+1} : \|x\| = 1\},$$

equipped with the metric (of constant sectional curvature 1) induced from ambient Euclidean space $\mathbb{E}^{k+1}$.

Theorem 2.4[9, Corollary 4.17]. Suppose $\varphi\colon (M,g) \to \mathbb{S}^n$, and $\tilde{\varphi}\colon (M,g) \to \mathbb{E}^{n+1}$ is the composition of φ with the inclusion map $\mathbb{S}^n \hookrightarrow \mathbb{E}^{n+1}$. Then φ is a r-harmonic map precisely when:

$$trace_{\mathcal{V}} \nabla^2 \tilde{\varphi} + d\tilde{\varphi}(div\nu) = -r\|(d\varphi)^r\|^2 \tilde{\varphi},$$

where $\mathcal{V} = \nu_{r-1}(\tilde{\varphi}) = \nu_{r-1}(\varphi)$.

Remark 2.5 When $r = 1$ the divergence term in Theorem vanishes, and the twisted trace reduces to the Laplace-Beltrami operator, which is linear. However this is no longer the case when $r > 1$.

The most familiar example to test drive Theorem is probably the following.

Example 2.6 [9, Example 4.19]. The Hopf map $\varphi\colon \mathbb{S}^3 \to \mathbb{S}^2$ is defined:

$$\varphi(u,v) = (2\bar{u}v, |v|^2 - |u|^2), \quad (u,v) \in \mathbb{C}^2,\ |u|^2 + |v|^2 = 1.$$

Then φ is 1-harmonic and 2-harmonic with:

$$\varepsilon_1(\varphi) = 8, \qquad \varepsilon_2(\varphi) = 16, \qquad div\nu_1(\varphi) = 0.$$

The natural generalisation of the Hopf map to higher dimensions is the Hopf fibration $\varphi\colon \mathbb{S}^{2n+1} \to \mathbb{C}P^n$, defined for all unit vectors $(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ by:

$$\varphi(z_1, \ldots, z_{n+1}) = [z_1, \ldots, z_{n+1}],$$

where the square brackets denote homogeneous coordinates. The following result is a special case of Theorem below.

Theorem 2.7 The Hopf fibration φ is a r-harmonic map for all $r = 1, \ldots, 2n$, with:

$$\varepsilon_r(\varphi) = \binom{2n}{r} \quad \text{(binomial coefficient)}, \quad div\nu_{r-1}(\varphi) = 0.$$

Remark 2.8 The r-power energy densities in Theorem when $n = 1$ differ from those of Example , since $\mathbb{S}^2$ is only homothetically equivalent to $\mathbb{C}P^1$. Furthermore, the fact that $\varepsilon_1(\varphi) < \varepsilon_2(\varphi)$ in Example whereas $\varepsilon_1(\varphi) > \varepsilon_2(\varphi)$ in Theorem shows that the relative values of the higher-power energies can be changed by homotheties of the codomain.

The Hopf fibration is an example of a Riemannian submersion, and the following theorem generalises a well-known result for harmonic maps.

Theorem 2.9 Suppose $\varphi\colon (M,g) \to (N,h)$ is a Riemannian submersion. Then for all $r = 1, \ldots, n$ we have:

$$\varepsilon_r(\varphi) = \binom{n}{r},$$

and the following are equivalent:

i) φ is a r-harmonic map;

ii) $div\nu_{r-1}(\varphi) = 0$ and $r \geq 2$;
iii) φ has minimal fibres.

Proof. As usual, we write $TM = \mathcal{V} \oplus \mathcal{H}$ where $\mathcal{V} = \ker d\varphi$ and $\mathcal{H} = \mathcal{V}^{\perp}$. We denote by $\pi_{\mathcal{V}}\colon TM \to \mathcal{V}$ and $\pi_{\mathcal{H}}\colon TM \to \mathcal{H}$ the projection morphisms. From (3), for all $X, Y \in \mathfrak{X}(M)$ we have:

$$\begin{aligned} g(\alpha(X), Y) &= h(d\varphi(X), d\varphi(Y)) = h(d\varphi \circ \pi_{\mathcal{H}}(X), d\varphi \circ \pi_{\mathcal{H}}(Y)) \\ &= g(\pi_{\mathcal{H}}(X), \pi_{\mathcal{H}}(Y)), \quad \text{since } \varphi \text{ is a Riemannian submersion} \\ &= g(\pi_{\mathcal{H}}(X), Y). \end{aligned}$$

Therefore $\alpha = \pi_{\mathcal{H}}$. Since $\dim \mathcal{H} = n$ the expression for $\varepsilon_r(\varepsilon)$ as the r-th elementary symmetric polynomial in the eigenvalues of $\pi_{\mathcal{H}}$ is immediate. It then follows from (2) that:

$$\begin{aligned} \nu_{r-1}(\varphi) &= \binom{n}{r-1} I - \binom{n}{r-2}\pi_{\mathcal{H}} + \cdots + (-1)^{r-2}\binom{n}{1}\pi_{\mathcal{H}} + (-1)^{r-1}\pi_{\mathcal{H}} \\ &= \binom{n}{r-1} I - \binom{n-1}{r-2}\pi_{\mathcal{H}}, \end{aligned}$$

by the Newton-Girard identity (1), with the understanding that the term involving $\pi_{\mathcal{H}}$ comes into play only if $r \geq 2$.

Now choose local orthonormal frames $\{H_i : 1 \leq i \leq n\}$ for $\mathcal{H}$ and $\{V_j : 1 \leq j \leq m-n\}$ for $\mathcal{V}$. Then abbreviating $\nu = \nu_{r-1}(\varphi)$, and using the fact that $\nabla d\varphi(\mathcal{H}, \mathcal{H}) = 0$ for a Riemannian submersion, it follows from (5) that (summing over repeated indices):

$$trace_{\mathcal{V}} \nabla d\varphi = \binom{n}{r-1} \nabla d\varphi(V_j, V_j) = (n-m)\binom{n}{r-1} d\varphi(H_{\varphi}),$$

where H_{φ} is the mean curvature of the fibres of φ. Furthermore:

$$\begin{aligned} div\pi_{\mathcal{H}} &= \nabla\pi_{\mathcal{H}}(H_i, H_i) + \nabla\pi_{\mathcal{H}}(V_j, V_j) \\ &= \pi_{\mathcal{V}}(\nabla H_i H_i) - \pi_{\mathcal{H}}(\nabla V_j V_j) \\ &= (n-m)H_{\varphi}, \end{aligned}$$

since the vertical component of $\nabla\colon \Gamma(\mathcal{H}) \times \Gamma(\mathcal{H}) \to \mathfrak{X}(M)$ is skew-symmetric [7]. Hence:

$$div\nu = (m-n)\binom{n-1}{r-2} H_{\varphi}.$$

From (4) we therefore obtain:

$$\tau_r(\varphi) = (n-m)\left(\binom{n}{r-1} - \binom{n-1}{r-2}\right) d\varphi(H_{\varphi}) = (n-m)\binom{n-1}{r-1} d\varphi(H_{\varphi}).$$

Since H_{φ} is horizontal it follows that φ is r-harmonic if and only if $H_{\varphi} = 0$, which also coincides with $div\nu = 0$ when $r \geq 2$.

Remark 2.10 Theorem shows that for Riemannian submersions there is no significant difference between the variational behaviour of the non-trivial higher-power energies.

The following result is "dual" to Theorem , and exhibits similar features. The proof is comparitively straightforward.

Theorem 2.11 Suppose $\varphi\colon (M,g) \to (N,h)$ is a Riemannian (ie. isometric) immersion. Then for all $r = 1,\dots,m$ we have:

$$\varepsilon_r(\varphi) = \binom{m}{r}, \quad div\nu_{r-1}(\varphi) = 0,$$

and φ is a r-harmonic map if and only if φ is a minimal immersion.

Proof. We have $\alpha = I$, hence:

$$\begin{aligned}\mathcal{V} = \nu_{r-1}(\varphi) &= \Big(\binom{m}{r-1} - \binom{m}{r-2} + \cdots + (-1)^{r-1}\Big)I \\ &= \binom{m-1}{r-1}I,\end{aligned}$$

by the Newton-Girard identity (**??**). Therefore $div\nu = 0$, and:

$$trace_{\mathcal{V}}\nabla d\varphi = \binom{m-1}{r-1}\tau(\varphi) = m\binom{m-1}{r-1}H_\varphi,$$

where H_φ is the mean curvature of φ.

Remark 2.12 Setting $m = n$ in either of Theorems or shows that the isometries of a Riemannian manifold are r-power harmonic maps for all r, as we would expect, and all their Newton tensors are divergence-free.

3. Higher-power harmonic sections

In differential geometry, many mappings of interest are sections of fibre bundles, and to treat them simply as maps ignores this extra structure. We therefore "tweak" the ideas of Section in a couple of ways, the most obvious of which is to restrict the calculus of variations to the submanifold of sections. However we would also like to measure the deviation of a section from "horizontality" (given a suitable definition of this concept), particularly in situations where no "horizontal" sections exist. This requires tweaking the higher-power energy functionals themselves.

To be more specific, suppose initially that $\pi\colon (P,k) \to (M,g)$ is a smooth submersion of Riemannian manifolds (not necessarily a Riemannian submersion, or a fibre bundle). We use the metric of P to make the following decomposition:

$$TP = \mathcal{V} \oplus \mathcal{H}, \tag{6}$$

where $\mathcal{V} = \ker d\pi$ and $\mathcal{H} = \mathcal{V}^\perp$. Now given a smooth section $\sigma\colon M \to P$ we define its *vertical derivative* $d^v\sigma$ by:

$$d^v\sigma(X) = \pi_{\mathcal{V}}(d\sigma(X)),$$

for all $X \in \mathfrak{X}(M)$, where $\pi_{\mathcal{V}}\colon TP \to \mathcal{V}$ is projection with respect to the splitting (6). The vertical derivative is a 1-form on M with values in the pullback bundle $\sigma^{-1}\mathcal{V} \to M$. *The vertical first fundamental tensor* (or *vertical Cauchy-Green tensor*) α^v of σ is then defined:

$$g(\alpha^v(X), Y) = k(d^v\sigma(X), d^v\sigma(Y)), \tag{7}$$

for all $X, Y \in \mathfrak{X}(M)$.

We now apply the algebraic apparatus of Section , defining first the elementary invariants of α^v, which we denote by $\varepsilon_r^v(\sigma)\colon M \to \mathbb{R}$ for $r = 1, \ldots, m$. Then:

$$\varepsilon_1^v(\sigma) = \|d^v\sigma\|^2, \ \ldots, \ \varepsilon_r^v(\sigma) = \|d^v\sigma \wedge \cdots \wedge d^v\sigma\|^2.$$

The first elementary invariant is familiar from the theory of harmonic sections [2]; it vanishes identically precisely when σ is horizontal in the sense that $d\sigma(TM) \subset \mathbb{H}$. In general, $\varepsilon_r^v(\sigma)$ vanishes at points where the rank of $d^v\sigma$ drops below r. The r-*power vertical energy* of σ is then defined:

$$\mathbb{E}_r^v(\sigma) = \frac{1}{2}\int_M \varepsilon_r^v(\sigma) vol(g), \quad r = 1, \ldots, m.$$

This functional is trivial if r exceeds the dimension of the fibres of π, which can only happen if $\dim(P) < 2\dim(M)$.

We also construct the Newton tensors of α^v, which we denote by $\nu_r^v(\sigma)$ and refer to as the *vertical Newton tensors* of σ (to distinguish them from the Newton tensors of σ as defined in Section , which are still available). These are defined for $r = 0, \ldots, m$, and are symmetric $(1,1)$-tensors on M, with $\nu_0^v(\sigma) = I$ and $\nu_m^v(\sigma) = 0$. The r-*power vertical tension fields* of σ are then defined for $r = 1, \ldots, m$ by:

$$\tau_r^v(\sigma) = trace\nabla^v(d^v\sigma \circ \nu) = trace_{\mathcal{V}}\nabla^v d^v\sigma + d^v\sigma(div\nu),$$

where $\mathcal{V} = \nu_{r-1}^v(\sigma)$ and ∇^v is the $\mathcal{V}$-component of the Levi-Civita connection of k. So $\tau_r^v(\sigma)$ is a section of $\sigma^{-1}\mathcal{V} \to M$, which simplifies if $\nu_{r-1}^v(\sigma)$ is solenoidal.

Theorem 3.1[9, Theorem 3.6]. Suppose that π has totally geodesic fibres. If σ_t is a smooth variation of σ through sections of π then:

$$\frac{d}{dt}\Big|_{t=0} \mathbb{E}_r^v(\sigma_t) = -\int_M k\Big(\tau_r^v(\sigma), \frac{\partial\sigma_t}{\partial t}\Big|_{t=0}\Big) vol(g).$$

Theorem leads to the following definition.

Definition 3.2 A smooth section σ of a submersion with totally geodesic fibres is a r-*power harmonic section*, or simply r-*harmonic section*, if $\tau_r^v(\sigma) = 0$, for $r = 1, \ldots, m$.

Remark 3.3 It is possible to remove the condition that π has totally geodesic fibres. However the Euler-Lagrange equations then become more complicated, and are no longer characterised by the vanishing of the higher-power vertical tension fields.

To demonstrate this machinery in action, we apply it first to the familiar case of a vector bundle $\pi\colon \mathbb{E} \to (M, g)$. Suppose, as is often the case, that π has a linear connection ∇ and holonomy-invariant fibre metric $< *, * >$; ie. π is a *Riemannian vector bundle.* A Riemannian metric k on $\mathbb{E}$ may then be obtained from the “Kaluza-Klein” construction (otherwise said, k is a generalised Sasaki metric), with respect to which π has totally geodesic fibres. Then:

$$\varepsilon_r^v(\sigma) = \|(\nabla\sigma)^r\|^2 = \|\nabla\sigma \wedge \cdots \wedge \nabla\sigma\|^2.$$

The condition for $\varepsilon_r^v(\sigma)$ to vanish at $x \in M$ is therefore that the rank of the $\mathbb{E}$-valued 1-form $\nabla\sigma$ on M drops below r at x. If this happens everywhere then we say that σ is r-*parallel*, generalising the familiar notion of a parallel section, which we recover when $r = 1$. Thus we see very clearly what our various measurements of "deviation from horizontality" mean in this context.

Under suitable (natural) identifications, the higher-power vertical tension fields may be viewed as operators $\mathcal{T}_r \colon \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ defined:

$$\mathcal{T}_r(\sigma) = trace_{\mathcal{V}} \nabla^2 \sigma + \nabla\, div \mathcal{V}\, \sigma, \qquad r = 1, \ldots, m, \tag{8}$$

where $\nu = \nu^v_{r-1}(\sigma)$ [9, Theorem 4.7]. (Note that the second covariant derivative $\nabla^2\sigma$ is built out of the connection in $\mathcal{E}$ *and* the Levi-Civita connection of g.) When $r = 1$ this reduces to the rough Laplacian; however $\mathcal{T}_r$ becomes non-linear if $r > 1$. Then, since π has totally geodesic fibres, σ is a r-harmonic section of $\mathcal{E}$ if and only if $\mathcal{T}_r(\sigma) = 0$.

The problematic feature of higher-power harmonic sections of vector bundles (and *a fortiori* sections that are higher-power harmonic maps) is the following rigidity theorem, already known in the "classic" case $r = 1$ [6].

Theorem 3.4[9, Theorem 4.5]. Suppose that σ is a section of a Riemannian vector bundle with compact base.

(i) σ is a r-harmonic section if and only if σ is r-parallel.

(ii) σ is a r-harmonic map if and only if σ is parallel.

To sidestep the ramifications of Theorem we turn to a closely related class of bundles, the associated (unit) sphere bundles $\mathcal{S} \to M$ defined:

$$\mathcal{S} = \{v \in \mathcal{E} :< v, v >= 1\}.$$

We equip with the restriction of the Kaluza-Klein metric of $\mathcal{E}$ (which coincides with the metric obtained from the Kaluza-Klein construction applied to $\mathcal{S}$). The following result is analogous to, and in fact generalises, the characterisation of higher-power harmonic maps into spheres (Theorem). It is already well-known when $r = 1$ [11].

Theorem 3.5 [9, Theorem 4.13]. The r-harmonic sections σ of are characterised by the equation:

$$\mathcal{T}_r(\sigma) = -r\|(\nabla\sigma)^r\|^2\sigma.$$

To illustrate these ideas, suppose that $\mathcal{E} = TM$ and k is the Sasaki metric. Then $\mathcal{S} = UM$, the unit tangent bundle, and sections of $\mathcal{S}$ are unit vector fields on M. For a concrete example, let $M = \mathbb{S}^{2n+1}$ and let σ be the Hopf vector field, defined for all unit vectors $z \in \mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ by:

$$\sigma(z) = iz,$$

where $i = \sqrt{-1}$.

Theorem 3.6 The Hopf vector field σ on $\mathbb{S}^{2n+1}$ is a r-harmonic section of the unit tangent bundle for all $r = 1, \ldots, 2n$, with:

$$\varepsilon^v_r(\sigma) = \binom{2n}{r}, \quad div\nu^v_{r-1}(\sigma) = 0.$$

Proof. The natural identification of the tangent spaces of $\mathbb{S}^{2n+1}$ with their tangent spaces is isometric and the inverse sends $d^v\sigma$ to $\nabla\sigma$. It then follows from (7) that $\alpha^v(\sigma) = 0$, since the integral curves of σ are geodesics. Furthermore, if $\{H_k : k = 1, \ldots, 2n\}$ is a local orthonormal horizontal frame field in $\mathbb{S}^{2n+1}$ (with respect to the Hopf fibration) and H is a horizontal tangent vector then (summing over k):

$$\begin{aligned}\alpha^v(H) &= g(\alpha^v(H), H_k)H_k + g(\alpha^v(H), \sigma)\sigma \\ &= g\big(\nabla H\sigma, \nabla H_k\sigma\big)H_k + g\big(\nabla H\sigma, \nabla\sigma\sigma\big)\sigma, \quad \text{by (7)} \\ &= g\big(\nabla H\sigma, \nabla H_k\sigma\big)H_k,\end{aligned}$$

again using the fact that σ is geodesic. Now the action of $\nabla\sigma$ on horizontal tangent vectors is simply multiplication by i, which is isometric. Hence:

$$\alpha^v(H) = g(iH, iH_k)H_k = g(H, H_k)H_k = H.$$

We conclude that α^v is orthogonal projection onto the horizontal distribution of the Hopf fibration.

It follows from the first part of the proof of Theorem that α^v coincides with the first fundamental tensor of the Hopf fibration. Borrowing some of the subsequent calculations from that proof then gives us:

$$\varepsilon^v_r(\sigma) = \binom{2n}{r}, \quad \nu(\sigma) = \binom{2n}{r-1}\sigma, \quad \nu(H) = \binom{2n-1}{r-1}H,$$

where $\nu = \nu^v_{r-1}(\sigma)$, since ν is also the $(r-1)$-st Newton tensor of the Hopf fibration. Then $div\nu = 0$ by Theorem , and from (8) the tension operators of $T\mathbb{S}^{2n+1}$ act on σ as follows:

$$\begin{aligned}\mathcal{T}_r(\sigma) &= trace_{\mathcal{V}}\nabla^2\sigma = \nabla^2\mathcal{V}H_kH_k\sigma + \nabla^2\mathcal{V}\sigma\sigma\sigma \\ &= \binom{2n-1}{r-1}\nabla^2 H_kH_k\sigma + \binom{2n}{r-1}\nabla^2\sigma\sigma\sigma.\end{aligned}$$

The second covariant derivatives can be readily evaluated since σ is a spherical Killing field, using the following well-known curvature identity:

$$\nabla^2 YZ\sigma = -R(\sigma, Y)Z = g(\sigma, Z)Y - g(Y, Z)\sigma.$$

Hence:

$$\mathcal{T}_r(\sigma) = -2n\binom{2n-1}{r-1}\sigma = -r\binom{2n}{r}\sigma = -r\,\varepsilon^v_r(\sigma)\sigma = -r\|(\nabla\sigma)^r\|^2\sigma,$$

and it follows from Theorem that σ is a r-power harmonic section of the unit sphere bundle, as claimed.

Remark 3.7 The covariant derivative $\nabla\sigma$ of the Hopf vector field σ has rank $2n$. It therefore follows from Theorem that σ is *not* a r-harmonic section of the full tangent bundle $T\mathbb{S}^{2n+1}$ for any $r = 1, \ldots, 2n$. However σ is a $(2n+1)$-harmonic section of $T\mathbb{S}^{2n+1}$, with $\mathbb{E}^v_{2n+1}(\sigma) = 0$.

Remark 3.8 The evident similarity between Theorem and Theorem shows that the first order variational theories of higher-power energy of the Hopf fibration and higher-power vertical energy of the Hopf vector field are essentially the same. However, there are known differences in the second order theories (ie. stability vs. instability) when $r = 1$ [4, 12], and it is an interesting and as yet unanswered question whether these persist for higher powers.

4. 3**-dimensional Lie groups**

So far, none of our examples have revealed any discrepancies between the various higher-power energies of maps, or vertical higher-power energies of sections. To find some examples where the higher-power (vertical) energies behave differently we consider a wider class of unit vector fields, generalising Theorem when $n = 1$. Let M be a 3-dimensional Lie group, which we relabel as G, equipped with a left-invariant Riemannian metric g. The algebraic and geometric classification of all such (G, g) was given by Milnor in [5], and we begin with a brief review.

When an orientation for G is chosen, g defines a vector cross product $\times$ on the Lie algebra $\mathfrak{g}$. *The Lie structure map* is the linear map $L\colon \mathfrak{g} \to \mathfrak{g}$ characterised by:
L(a $\times$ b)=[a,b], for all $\alpha, \beta \in \mathfrak{g}$.
Then L is self-adjoint if and only if G is unimodular, which henceforward for simplicity we assume to be the case. So if $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of L, and $\sigma_1, \sigma_2, \sigma_3$ is a positively-oriented orthonormal eigenbasis of $\mathfrak{g}$, then:
$[\sigma_i, \sigma_j] = \lambda_k \sigma_k$,
for any even permutation (i, j, k) of $(1, 2, 3)$. We therefore refer to the λ_i as the *principal structure constants* of (G, g), and the σ_i as *principal structure directions*. Their dependence on orientation means that the principal structure constants are only determined up to sign. Furthermore if any of the λ_i have multiplicity then the principal structure directions are not unique.

There are precisely six possibilities for the relative signs of the principal structure constants (including 0), and these classify $\mathfrak{g}$ algebraically. The six classes are:

$$\mathfrak{su}(2), \qquad \mathfrak{sl}(2), \qquad \mathfrak{e}(2), \qquad \mathfrak{e}(1,1), \qquad \mathfrak{nil}, \qquad \mathfrak{a},$$

where $\mathfrak{e}(2)$ (resp. $\mathfrak{e}(1,1)$) is the Lie algebra of the isometry group of the Euclidean (resp. Minkowskian) plane, $\mathfrak{nil}$ is the 3-dimensional Heisenberg algebra and $\mathfrak{a}$ is the 3-dimensional abelian Lie algebra. The first two are simple, and the others are solvable.

The geometry of (G, g) is facilitated by the *Milnor map* $M\colon \mathfrak{g} \to \mathfrak{g}$ defined:

$$M = \tfrac{1}{2} trace(L) I - L.$$

(This looks similar to the first Newton tensor of L, but for the factor $1/2$.) The principal structure directions are therefore also eigenvectors of M, and we refer to the eigenvalues μ_1, μ_2, μ_3 as the *Milnor numbers* of (G, g). They are again only determined up to sign, and in terms of the principal structure constants are given by:

$$\mu_i = \tfrac{1}{2}(\lambda_j + \lambda_k - \lambda_i), \quad \{i, j, k\} = \{1, 2, 3\}. \tag{9}$$

Most remarkably, the Ricci curvature Ric of (G, g), when viewed as a $(1, 1)$-tensor, is essentially the 2-nd Newton tensor of M:

$$Ric = 2\nu_2(M).$$

It follows that the principal structure directions are also principal Ricci directions, and the principal Ricci curvatures ρ_i (ie. the eigenvalues of Ric) are:

$$\rho_i = 2\mu_j \mu_k, \quad \{i, j, k\} = \{1, 2, 3\}.$$

Now let σ be a left-invariant unit vector field on G, regarded as a section of the unit tangent bundle $UG \to G$; ie. the unit sphere bundle of TG. We begin with a special case.

Theorem 4.1[9, Theorems 5.12 and 5.13]. Let (G, g) be a 3-dimensional unimodular Lie group with left-invariant metric. The principal structure directions are 1-harmonic *and* 2-harmonic sections of UG, with:

$$\varepsilon_1^v(\sigma_i) = \mu_j^2 + \mu_k^2, \qquad \varepsilon_2^v(\sigma_i) = \mu_j^2\mu_k^2, \qquad div\nu_1^v(\sigma_i) = 0,$$

where $\{i, j, k\} = \{1, 2, 3\}$.

Remark 4.2 Both vertical energy densities are independent of orientation, as they should be, and $\varepsilon_2^v(\sigma_i) = \frac{1}{4}\rho_i^2$.

Example 4.3 Let $G = SU(2)$. The principal structure constants for the Lie algebra $\mathfrak{su}(2)$ all have the same sign, which by choice of orientation may be assumed positive. The spherical metric then corresponds to the special case $\lambda_1 = \lambda_2 = \lambda_3 = 2$. All left-invariant unit vector fields σ are therefore principal directions, and congruent to the standard Hopf field. By (9) the Milnor numbers are $\mu_1 = \mu_2 = \mu_3 = 1$. So σ is 1-harmonic and 2-harmonic, with $\varepsilon_1^v(\sigma) = 2$ and $\varepsilon_2^v(\sigma) = 1$, in agreement with Theorem .

Example 4.4[9, Example 5.3]. Let $G = PSL(2, \mathbb{R})$ which we identify with the unit tangent bundle of the hyperbolic plane:

$$U\mathbb{H}^2 = \{(x, y, \theta) : x \in \mathbb{R},\ y \in \mathbb{R}^+,\ \theta \in \mathbb{S}^1\}.$$

We endow G with the Sasaki metric g, which is left-invariant and may be expressed as follows:

$$ds^2 = \frac{1}{y^2}\big(dx^2 + dy^2 + (dx + yd\theta)^2\big).$$

Consider the left-invariant unit vector fields $\sigma_1, \sigma_2, \sigma_3$ on G defined:

$$\sigma_1 = y\cos\theta\,\partial_x + y\sin\theta\,\partial_y - \cos\theta\,\partial_\theta, \quad \sigma_2 = -y\sin\theta\,\partial_x + y\cos\theta\,\partial_y + \sin\theta\,\partial_\theta, \quad \sigma_3 = \partial_\theta.$$

Standard computational techniques for Lie brackets of vector fields yield:

$$[\sigma_1, \sigma_2] = -\sigma_3, \qquad [\sigma_2, \sigma_3] = \sigma_1, \qquad [\sigma_3, \sigma_1] = \sigma_2.$$

Therefore the σ_i are principal structure directions, with:

$$\lambda_1 = \lambda_2 = 1, \qquad \lambda_3 = -1.$$

Hence $\mu_1 = \mu_2 = -1/2$ and $\mu_3 = 3/2$ by (9). It then follows from Theorem that:

- σ_3 is 1-harmonic and 2-harmonic, with:

$$\varepsilon_1^v(\sigma_3) = 1/2, \qquad \varepsilon_2^v(\sigma_3) = 1/16.$$

- For all $t \in [0, 2\pi)$ the unit vector field σ_t defined:

$$\begin{aligned}\sigma_t &= (\cos t)\sigma_1 + (\sin t)\sigma_2\\ &= y\cos(\theta + t)\,\partial_x + y\sin(\theta + t)\,\partial_y - \cos(\theta - t)\,\partial_\theta\end{aligned}$$

  is 1-harmonic and 2-harmonic, with:

$$\varepsilon_1^v(\sigma_t) = 5/2, \qquad \varepsilon_2^v(\sigma_t) = 9/16.$$

Remark 4.5 A comparison of the vertical energy densities in Example suggests that σ_t is unstable with respect to both vertical 1-energy and 2-energy, and it is possible to show that this is indeed the case by explicity constructing an $\mathbb{E}^v_r$-decreasing variation from σ_t to σ_3 for $r = 1, 2$. We also suspect that σ_3 is stable, and possibly an absolute minimum of both vertical energies, although this has not yet been verified.

We now come to the general case. Notation-wise, we denote by M^2 and Ric^2 the twice-iterated Milnor and Ricci tensors:

$$M^2 = M \circ M, \qquad Ric^2 = Ric \circ Ric.$$

Theorem 4.6[9, Theorems 5.12 and 5.13]. Let σ be a left-invariant unit vector field on a 3-dimensional unimodular Lie group G with left-invariant metric.

i) σ is a 1-harmonic section of UG if and only if σ is an eigenvector of M^2.

ii) σ is a 2-harmonic section of UG if and only if σ is an eigenvector of Ric^2.

Remark 4.7 The non-unimodular case was also addressed in [8].

Since the principal structure directions are eigenvectors of both M and Ric, the eigenspaces of M^2 and Ric^2 are direct sums of those of L. The admissible configurations are indicated in Figure 9, which shows the possible subsets (orange) of the unit sphere of $\mathfrak{g}$ (blue) in which higher-power harmonic left-invariant unit vector fields can lie.

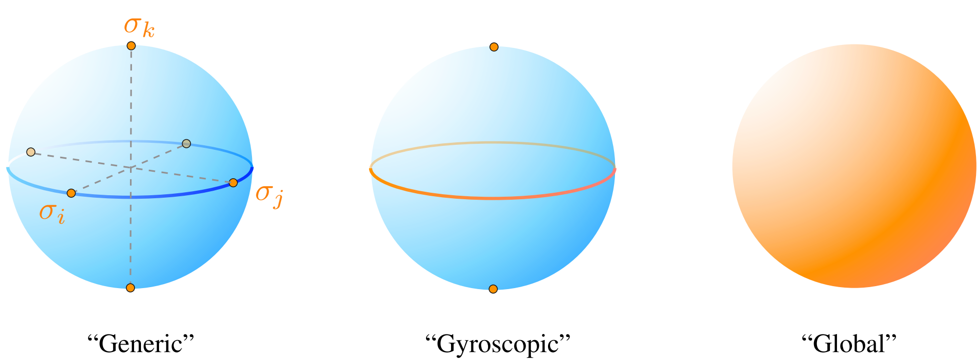

Figure 9: Admissible configurations

The "gyroscopic" and "global" configurations can occur when some or all of the principal structure constants are distinct, and may differ according to the power of the energy, as illustrated by the following examples.

Example 4.8 Suppose $\mathfrak{g} = \mathfrak{e}(2)$. For this Lie algebra precisely one principal structure constant vanishes, and the other two have the same sign. An exceptional case is when the two non-vanishing λ_i are equal; the corresponding metric is then flat (in fact, these are the only flat metrics on any non-abelian 3-dimensional unimodular Lie group). For these metrics the 1-harmonic left-invariant unit vector fields are gyroscopic, and the 2-harmonic fields are global.

Example 4.9 Suppose $\mathfrak{g} = \mathfrak{sl}(2)$. The principal structure constants for this Lie algebra are all non-zero, and precisely two have the same sign, which by choice of orientation may be assumed positive. If labelling is chosen so that $\lambda_1 \geq \lambda_2, \lambda_3$

then an exceptional case is $\lambda_1 = \lambda_2 - \lambda_3$ or $\lambda_1 = \lambda_3 - \lambda_2$, depending on which of λ_2, λ_3 is negative. (Note that this is not the exceptional case in Example , where two principal structure constants are equal.) The left-invariant unit vector fields that are 1-harmonic with respect to these metrics are generic, whereas the 2-harmonic fields are gyroscopic.

Example 4.10 Suppose $\mathfrak{g} = \mathfrak{su}(2)$. By choice of orientation and labelling it may be assumed that $\lambda_1, \lambda_2, \lambda_3 > 0$ with $\lambda_1 \geq \lambda_2, \lambda_3$. An exceptional case is $\lambda_1 = \lambda_2 + \lambda_3$, $\lambda_2 \neq \lambda_3$, and for these metrics the 1-harmonic left-invariant unit vector fields are again generic, whereas the 2-harmonic fields are gyroscopic.

It's possible, but complicated, to classify the higher-power harmonic invariant unit vector fields for all possible left-invariant metrics on G [9, Theorem 5.16]. Instead we conclude with a few pertinent observations.

Corollary 4.11 [9, Corollary 5.15]. Let σ be a left-invariant unit vector field on a 3-dimensional unimodular Lie group G with left-invariant metric.

i) If σ is a 1-harmonic section of UG then σ is 2-harmonic.
ii) If σ is a 2-harmonic section but not 1-harmonic then $Ric(\sigma) = 0$, which is the case precisely when $\varepsilon_2^v(\sigma) = 0$.
iii) The only cases where σ is 2-harmonic but not 1-harmonic are $\mathfrak{g} = \mathfrak{su}(2)$, $\mathfrak{sl}(2)$ or $\mathfrak{e}(2)$ with the metrics described in Examples 4.8–4.10.

Remark 4.12 The vertical first Newton tensor ν of a 2-harmonic left-invariant unit vector field need not be solenoidal. For example, if $G = E(2)$ (the Euclidean group) then the infinitesimal translations are 1-harmonic and 2-harmonic, for all left-invariant metrics, but ν is only solenoidal with respect to the flat metrics, which render all left-invariant unit fields 2-harmonic (see Example 4.8).

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RICCI SOLITONS WITH CONCIRCULAR AND CONFORMAL KILLING POTENTIAL VECTOR FIELDS IN COMPLEX SASAKIAN MANIFOLDS

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ABSTRACT

Ricci solitons with concircular and conformal killing potential vector fields in complex Sasakian manifolds are investigated. In addition, it is shown that a Ricci soliton in complex Sasakian manifolds satisfying the conditions $\rho(U, X)R = 0$ and $\rho(V, X)R = 0$ is always expanding.

***K*eywords** Concircular, conformal killing, complex Sasakian manifolds, Ricci solitons

1. Introduction

The Riemannian geometry of contact manifolds are studied widely at last 60 years and could be divided into two parts: reel and complex. While there are rather than works on the Riemannian geometry of reel contact manifolds the complex contact manifolds are still infancy. One can think that is it possible to transfer all results from real contact manifolds to complex unchanged. But it is not possible, so the Riemannian geometry of complex contact manifolds should be studied independently.

Theory of complex contact manifolds was started by Kobayashi's paper [21] in 1959. Although this theory is older as real contact manifolds it does not interested to geometers like real contact manifolds. Further research was started again in the early 1980s by Ishihara and Konishi [16, 17]. Ishihara and Konishi gave Hermitian metric on complex contact manifolds and they showed that a complex contact manifold admits always a complex almost contact structure of class C^∞ [18]. Same authors presented their normality conditions for these manifolds and they also proved that when a complex contact manifold is normal underlying manifold is Kaehler. When a complex contact metric manifold is normal in this sense it is called IK-normal. IK-normality generates some restrictions for studying normal complex contact manifolds. The one example of complex contact manifolds odd-dimensional complex projective space is Kaehler and IK-normal. But while real Heisenberg group is the well-known example of normal real contact manifolds, since complex Heisenberg group is not Kaehler, it is not IK-normal. For overcoming this issue in 2000 Korkmaz gave a weaker definition for normality [23]. This definition can be considered complex version of Sasakian manifolds [14]. On the other hand there is no another example of normal complex contact manifolds except for complex Heisenberg group. Many years after especially Blair, Foreman and Korkmaz improved the Riemannian geometry of complex contact metric manifolds [2, 3, 4, 13, 14, 23]. Also complex contact manifolds have some different results from real case and they have good applications in optimal control of entanglements [20].

A special case of normal complex contact metric manifolds is a complex Sasakian manifold. Foreman gave the definition of the complex Sasakian manifold in 2000 [14]. In addition, Fetcu examined harmonic maps between complex Sasakian manifolds [10] and an adapted connection on a strict complex contact manifold. [11]. We have given the definition of the complex Sasakian manifold in accordance with the real Sasakian in [32]. We studied symmetry in complex Sasakian manifolds in [33].

The concept of Ricci solitons was introduced by R. Hamilton [15] in the mid 1980's. In 1993, Iwey studied Ricci solitons on compact three- manifolds [19]. In 2006, Cao studied geometry of Ricci solitons [9].

A Ricci soliton is a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M, g). A Ricci soliton is a triple (g, W, λ) with g a Riemannian metric, W a vector field, and λ a real scalar such that

$$\mathcal{L}_W g + 2\rho + 2\lambda g = 0$$

where $\mathcal{L}_W$ denotes the Lie derivative along W, ρ is the Ricci tensor. The vector field W is called the potential field.. Obviously, a trivial Ricci soliton is an Einstein metric with W zero or Killing. Thus, a Ricci soliton may be considered an apt generalization of an Einstein metric. The Ricci soliton is said to be shrinking, steady, and expanding accordingly as λ is negative, zero, and positive, respectively.

Fialkow introduced in [12] the notion of concircular vector fields on a Riemannian manifold. The concept of conformal Killing on a Riemannian manifold was introduced by Tachibana in [25]. Chen and Deshmukh first introduced and classified Ricci solitons with congruent vector fields on a Riemannian manifold in 2015 [7]. He classified Ricci solitons with concircular potential field on a Riemannian manifold [8].

2. Complex Almost Contact Metric Manifolds

The study of complex contact manifolds began with Shoshichi Kobayashi in 1959. In [21] Kobayashi gave the following definition;

Definition 2.1 Let M be a $(2n+1)-$complex dimensional complex manifold and $\mathcal{A} = \{\mathcal{O}, \mathcal{O}', ...\}$ be an open covering by coordinate neighbourhoods with following conditions:

- There is a holomoprhic 1-form on ω each $\mathcal{O}$ such that $\omega \wedge (d\omega)^n \neq 0$,

- There is a holomoprhic function $\lambda : \mathcal{O} \cap \mathcal{O}' \to \mathbb{C} \setminus \{0\}$ such that $\omega' = \lambda\omega$.

Then ω is called a complex contact form, and M is called a complex contact manifold.

The complex contact structure determines a non-integrable distribution $\mathcal{H}$ by the equation $\omega = 0$. $\mathcal{H}$ is called a horizontal subbundle.

A complex almost contact metric structure related to complex contact form was defined by Ishihara and Konishi in [16].

Definition 2.2 Let M be an odd complex dimensional complex manifold with almost complex structure J and Hermitian metric g and be covered by an open covering $\mathcal{A} = \{U_i\}$ consisting of coordinate neighborhoods. If the following conditions satisfy then the manifold is called a complex almost contact metric manifold:

1. In each $\mathcal{U}_i$ there exist 1-forms u and $v = u \circ J$, with dual vector fields U and $V = -JU$ and $(1, 1)$ tensor fields G and $H = GJ$ such that
$$H^2 = G^2 = -I + u \otimes U + v \otimes V$$
$$GJ = -JG, \quad GU = 0, \quad g(X, GY) = -g(GX, Y).$$
2. For a and b are functions on $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$ with $a^2 + b^2 = 1$ we have
$$\begin{aligned} u' &= au - bv, \quad v' = bu + av, \\ G' &= aG - bH, \quad H' = bG + aH. \end{aligned}$$

U and V vector fields determine a global vertical distribution $\mathcal{V}$ which is typically assumed to be integrable.Thus from Whitney sum we have
$$TM = \mathcal{H} \oplus \mathcal{V}$$
and so an arbitrary vector field X on M we can write
$$X = X_0 + u(X)U + v(X)V, \quad X_0 \in \mathcal{H}.$$
The local tensor fields G and H are related to du and dv by
$$\begin{aligned} du(X, Y) &= \widetilde{G}(X, Y) + (\sigma \wedge v)(X, Y) \\ dv(X, Y) &= \widetilde{H}(X, Y) - (\sigma \wedge u)(X, Y) \end{aligned}$$
where fundamental $2-$forms $\widetilde{G}$ and $\widetilde{H}$ are defined by $\widetilde{G}(X, Y) = g(X, GY)$, $\widetilde{H}(X, Y) = g(X, HY)$, and $\sigma(X) = g(\nabla_X U, V)$. σ is also called the Ishihara-Konishi connection [16, 17].

In complex almost contact geometry, we have two normality notions. One of is given by Ishihara and Konishi [17]. The authors defined the following two tensor
$$\begin{aligned} S(X, Y) &= [G, G](X, Y) + 2g(X, GY)U - 2g(X, HY)V \\ &\quad +2(v(Y)HX - v(X)HY) + \sigma(GY)HX \\ &\quad -\sigma(GX)HY + \sigma(X)GHY - \sigma(Y)GHX, \qquad (1) \\ T(X, Y) &= [H, H](X, Y) - 2g(X, GY)U + 2g(X, HY)V \\ &\quad +2(u(Y)GX - u(X)GY) + \sigma(HX)GY \\ &\quad -\sigma(HY)GX + \sigma(X)GHY - \sigma(Y)GHX. \qquad (2) \end{aligned}$$

In here

$$[G,G](X,Y)=(\nabla_{GX}G)Y-(\nabla_{GY}G)X-G(\nabla_X G)Y+G(\nabla_Y G)X$$

is the Nijenhuis torsion of G. If $S=T=0$, then a complex almost contact metric manifold is called normal. This type of normality has known as IK-normality. On a IK-normal complex contact metric manifold, we have $\nabla J=0$, that is the manifold is Kähler.

Ishihara and Konishi introduced a concept of normality, which forced the structure to be Kählerian and did not include some natural examples like the complex Heisneberg group. This led Korkmaz to define a weaker version of normality in [22], which included these examples. The an other difference, which makes this concept interesting, a normal complex contact metric manifold is not Kaehler.

Definition 2.3 A complex almost contact metric manifold is normal if

$S(X,Y)=T(X,Y)=0$ for all X,Y in $\mathcal{H}$, and

$S\left(X,U\right)=T\left(X,V\right)=0$ for all X .

In literature, we recall this kind of manifolds as *normal complex contact metric manifolds*.

Korkmaz proved following theorem which states normality condition.

Theorem 2.4 A complex almost contact metric manifold is normal if and only if for $X,Y,Z\in TM$

$$\begin{aligned}
g((\nabla_X G)Y,Z) &= \sigma(X)g(HY,Z)+v(X)d\sigma(GZ,GY)\\
&\quad -2v(X)g(HGY,Z)-u(Y)g(X,Z)\\
&\quad -v(Y)g(JX,Z)+u(Z)g(X,Y)+v(Z)g(JX,Y), \qquad (3)\\
g((\nabla_X H)Y,Z) &= -\sigma(X)g(GY,Z)-u(X)d\sigma(HZ,HY)\\
&\quad +2u(X)g(HGY,Z)+u(Y)g(JX,Z)\\
&\quad -v(Y)g(X,Z)+u(Z)g(X,JY)+v(Z)g(X,Y). \qquad (4)
\end{aligned}$$

It is follow from above Theorem that;

$$\begin{aligned}
g((\nabla_W J)Z,T) &= u(W)(d\sigma(T,GZ)-2g(HZ,T))\\
&\quad +v(W)(d\sigma(T,HZ)+2g(GZ,T)).
\end{aligned}$$

which shows a normal complex contact metric manifold is not Kähler. In addition, Blair and Mihai studied symmetry in complex contact geometry [4], homogeneity and local symmetry of complex (κ, ν)-spaces [3]. In 2017, we introduced complex $\eta-$Einstein normal complex contact metric manifolds as follow [30].

Definition 2.5 Let (M,G,H,J,U,V,u,v,g) be a normal complex contact metric manifold and $\eta=u-iv$. If for α and β smooth functions on M the Ricci tensor satisfies

$$\rho=\alpha g+\beta(u\otimes u+v\otimes v)$$

then M is called a complex $\eta-$Einstein.

3. Complex Sasakian Manifolds

A special case of normal complex contact metric manifolds is a complex Sasakian manifold. Foreman gave the definition of the complex Sasakian manifold in 2000 as follows [14].

Definition 3.1 A normal complex contact metric manifold whose complex contact structure is given by a global complex contact form, is called a complex Sasakian manifold.

Also Foreman obtained following result;

Theorem 3.2 Let $(M, G, H, J, U, V, u, v, g)$ be a normal complex contact metric manifold. If $\eta = u - iv$ is globally defined then $\sigma = 0$ [14].

We have given the definition of the complex Sasakian manifold in accordance with the real Sasakian manifold as follows [32].

Definition 3.3 Let $(M, G, H, J, U, V, u, v, g)$ be a normal complex contact metric manifold. If fundamental 2- forms $\widetilde{G}$ and $\widetilde{H}$ is defined by

$$\widetilde{G}(X,Y) = du(X,Y) \text{ and } \widetilde{H}(X,Y) = dv(X,Y)$$

then M is called a complex Sasakian manifold, where X, Y are vector fields on M.

We have given the version of the theorem found in the real Sasakian manifold as follows [32].

Theorem 3.4 Let M be a normal complex contact metric manifold. Then M is a complex Sasakian manifold if and only if

$$\begin{aligned}(\nabla_X G)Y &= -2v(X)HGY - u(Y)X - v(Y)JX \\ &+ g(X,Y)U + g(JX,Y)V \end{aligned} \quad (5)$$

$$\begin{aligned}(\nabla_X H)Y &= -2u(X)HGY + u(Y)JX - v(Y)X \\ &\quad -g(JX,Y)U + g(X,Y)V. \end{aligned} \quad (6)$$

So,we have

$$(\nabla_X J)Y = -2u(X)HY + 2v(X)GY$$

On a complex Sasakian manifold we get

$$\nabla_X U = -GX, \quad \nabla_X V = -HX. \quad (7)$$

Example 3.5 A complex Heisenberg group is an example of complex Sasakian manifolds.

We have studied on properties of Riemannian curvature tensor on complex Sasakian manifolds in [30]. Let M be a complex Sasakian manifold. Then we have,

$R(U,V)V = R(V,U)U = 0,$

$R(X,U)U = X + u(X)U + v(X)V,$

$R(X,V)V = X - u(X)U - v(X)V,$

$R(X,U)V = -3JX - 3u(X)V + 3v(X)U,$

$R(X,V)U = 0,$

$R(U,V)X = JX + u(X)V - v(X)U,$

$$\begin{aligned}R(X,Y)U &= v(X)JY - v(Y)JX + 2v(X)u(Y)V - 2v(Y)u(X)V \\ &+ u(Y)X - u(X)Y - 2g(JX,Y)V,\end{aligned}$$

$$\begin{aligned}R(X,Y)V &= 3u(X)JY - 3u(Y)JX - 2u(X)v(Y)U + 2u(Y)v(X)U \\ &+ v(Y)X - v(X)Y + 2g(JX,Y)U,\end{aligned}$$

$R(X,U)Y = -2v(Y)v(X)U + 2u(Y)v(X)V - g(Y,X)U + u(Y)X + g(JY,X)V,$

for $X, Y \in \Gamma(TM)$.
On a complex Sasakian manifold M, we showed the following properties.

$$\begin{aligned}
g(R(X,GX)GX,X) + g(R(X,HX)HX,X) + g(R(X,JX)JX,X) &= -6g\left(X,X\right),\\
g(R(X,GX)Y,GY) &= g(R(X,Y)X,Y) + g(R(X,GY)X,GY) - 2g(GX,Y)^2\\
&\quad - 4g(HX,Y)^2 - 2g(X,Y)^2 + 2g(X,X)g(Y,Y) - 4g(JX,Y)^2,\\
g(R(X,HX)Y,HY) &= g(R(X,Y)X,Y) + g(R(X,HY)X,HY) - 2g(HX,Y)^2\\
&\quad - 4g(GX,Y)^2 - 2g(X,Y)^2 + 2g(X,X)g(Y,Y) - 4g(JX,Y)^2,\\
g(R(X,HX)JX,GX) &= -g(R(X,HX)HX,X) - 4g(X,X)^2,\\
g(R(X,JX)HX,GX) &= g(R(X,JX)JX,X) - 2g(X,X)^2,\\
g(R(GX,HX)HX,GX) &= g(R(X,JX)JX,X),\\
g(R(GX,JX)JX,GX) &= g(R(X,HX)HX,X),\\
g(R(JX,JY)JY,JX) &= g(R(X,Y)Y,X),\\
g(R(X,Y)JX,JY) &= g(R(X,Y)Y,X) + 4g(X,GY)^2 + 4g(X,HY)^2,\\
g(R(Y,JX)JX,Y) &= g(R(X,JY)JY,X),\\
g(R(X,JY)JX,Y) &= g(R(X,JY)JY,X) + 4g(X,HY)^2 + 4g(X,GY)^2,\\
g(R(X,JX)JY,Y) &= -g(R(JX,JY)X,Y) - g(R(JY,X)JX,Y),
\end{aligned}$$

By considering above results we obtain following:
Theorem 3.6 For a unit horizontal vector X on M we have

$$K\left(X,GX\right)+K\left(X,HX\right)+K\left(X,JX\right) = 6. \tag{8}$$

Theorem 3.7 Let M be a complex Sasakian manifold and X be a unit horizontal vector field on M. Then, the sectional curvature K is given by

$$K(U,V) = 0 \quad \text{and} \quad K(X,U) = 1.$$

On complex Sasakian manifolds , we obtain following relations

$$\begin{aligned}
\rho(U,U) &= \rho(V,V) = 4n, \;\; \rho(U,V) = 0,\\
\rho(X,U) &= 4nu(X), \; \rho\left(X,V\right) = 4nv(X),\\
\rho(X,Y) &= \rho(GX,GY) + 4n\left(u(X)u(Y) + v(X)v(Y)\right),\\
\rho(X,Y) &= \rho(HX,HY) + 4n\left(u(X)u(Y) + v(X)v(Y)\right),
\end{aligned} \tag{9}$$

where X, Y are any vector fields on M.

4. Ricci Soliton

In this section we will examine Ricci solitons on a complex Sasakian manifold. Now, consider the real characteristic vector fields U and V of a complex Sasakian manifold.

$$\begin{aligned}
(L_U g)(Y,Z) &= g(\nabla_Y U, Z) + g(\nabla_Z U, Y)\\
&= g(-GY,Z) + g(-GZ,Y)\\
&= -g(GY,Z) + g(Z,GY)\\
&= 0,
\end{aligned}$$

Thus, U is a killing vector field. Similarly,

$$\begin{aligned}(L_V g)(Y,Z) =& g(\nabla_Y V, Z) + g(\nabla_Z V, Y)\\ =& g(-HY, Z) + g(-HZ, Y)\\ =& -g(HY, Z) + g(Z, HY) = 0.\end{aligned}$$

Hence, V is a killing vector field.
Let X_0 and Y_0 be two horizontal vector fields. Hence, we have

$$(L_U g)(X_0, Y_0) + 2\rho(X_0, Y_0) + 2\lambda g(X_0, Y_0) = 0.$$

Since U is killing vector field we get

$$\rho(X_0, Y_0) = -\lambda g(X_0, Y_0). \tag{10}$$

For any vector fields $X_0 = X - u(X)U - v(X)V$, $Y_0 = Y - u(X)U - v(X)V$ we obtain

$$\rho(X,Y) = -\lambda g(X,Y) + (4n+\lambda)(u(X)u(Y) + v(X)v(Y)),$$

$$\begin{aligned}\tau =& \sum_{i=1}^{4n}[-\lambda g(E_i,E_i) + (4n+\lambda)(u(E_i)u(E_i) + v(E_i)v(E_i))]\\ &- \lambda g(U,U) + (4n+\lambda)(u(U)u(U) + v(U)v(U))\\ &- \lambda g(V,V) + (4n+\lambda)(u(V)u(V) + v(V)v(V))\\ =& -4n\lambda + 8n.\end{aligned}$$

Theorem 4.1 If the real characteristic vector fields of a complex Sasakian manifold are U and V, then Ricci solitons (g, U, λ) and (g, V, λ) in a complex Sasakian manifold are trivial.
Let (g, W, λ) be a Ricci soliton. If W is a conformal killing vector field then we have

$$\begin{aligned}&2\mu g + 2\rho + 2\lambda g = 0\\ &\rho = -(\mu + \lambda)g.\end{aligned}$$

From Eq. 19 we get

$$\rho(X,Y) = 8ng(X,Y) - 4n(u(X)u(Y) + v(X)v(Y)).$$

Hence, M is a complex $\eta-$Einstein. A. Fialkow introduced in [12] the notion of concircular vector fields on a Riemannian manifold M as follow.

Definition 4.2 A vector ν on a Riemannian manifold M is called concircular if it satisfies $\nabla_X \nu = \gamma X$ for any vector X tangent to M, where ∇ is the Levi-Civita connection of M, and γ is a function. A concircular vector field satisfying $\nabla_X \nu = \gamma X$ is called a nontrivial if γ is non-constant.
A vector field ν is called concurrent if $\gamma = 1$.

Definition 4.3 A Ricci soliton (g, W, λ) on a Riemannian manifold (M^n, g) is said to have concircular (or concurrent) potential field if its potential field W is a concircular (or concurrent) vector field. We call the vector field W the potential field of the Ricci soliton.

Theorem 4.4 A Ricci soliton with concircular vector field in complex Sasakian manifolds

$i)$ If $\gamma = -4n$ then the Ricci soliton is steady,

$ii)$ If $\gamma > -4n$ then the Ricci soliton is expanding

$iii)$ If $\gamma < -4n$ then the Ricci soliton is shrinking.

Corollary 4.5 Let M be a complex Sasakian manifold. If M is a Ricci soliton with concurrent vector field then the Ricci soliton is shrinking with $\lambda = -(4n+1)$.

Theorem 4.6 Let M be a complex Sasakian manifold with (g, W, λ) a Ricci soliton. If W is a concircular vector field then M is Einstein.

The concept of conformal Killing on a Riemannian manifolds was introduced by Tachibana in [25].

Definition 4.7 A vector field W on a Riemannian manifold is called conformal Killing vector if the Lie derivative satisfy

$$\mathcal{L}_W g = \mu g$$

for a function μ.

- If $\mu = 0 \Longrightarrow$ the vector field is a Killing vector
- If $\mu = 2c$ is constant the vector field is homothetic.

Theorem 4.8 Ricci soliton in complex Sasakian manifolds with conformal Killing vector field

$i)$ If $\mu = -8n$ then the Ricci soliton is steady,

$ii)$ If $\mu > -8n$ then the Ricci soliton is expanding

$iii)$ If $\mu < -8n$ then the Ricci soliton is shrinking.

Corollary 4.9 If a conformal Killing vector field W is the potential vector field of the Ricci soliton in complex Sasakian manifolds then W is homothetic vector field.

Theorem 4.10 Let M be a complex Sasakian manifold and (g, W, λ) a Ricci soliton. If W is conformal killing vector field then M is an Einstein.

Theorem 4.11 A Ricci soliton in complex Sasakian manifolds satisfying $\rho(U, X)R = 0$ and $\rho(V, X)R = 0$ is always expanding.

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DIFFERENTIAL GEOMETRY OF UMBRELLA MATRICES

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ABSTRACT

In this paper, studies on umbrella matrices will be discussed. Erdoğan Esin's work with umbrella matrices will be mentioned.

Keywords Umbrella matrix, Lie group, Infinitesimal motion, Orthogonal matrix, Darboux matrix.

1. Introduction

This study has been prepared on umbrella matrices and consists of three sections.

In the first section Ertugrul Özdamar's doctoral thesis named *"Lie Group of Umbrella Matrices and Differential Geometry"*, who is the first person to work on umbrella matrices in the literature will be discussed. In this section, after the definition of umbrella matrices in real and complex spaces, the Lie Group structure of umbrella matrices and the Lie algebraic structure of the Lie group is examined.

In the second section, Nuri Kuruoğlu's work, *"On The Lie Group of Umbrella Matrices"*, is discussed. This section shows that the umbrella matrix group is a subgroup of $GL(n, \mathbb{R})$. Also, Double umbrella matrices and Lie group are mentioned.

In the third section, Erdoğan Esin's doctoral dissertation titled *"Motions Along a Curve and Umbrella Matrices"* was written on a problem about the kinematics of umbrella matrices in correspondence between Hasan Hilmi Hacısalihoğlu and Oene Bottema was examined. This review was handled by Erdogan Esin's work *"Umbrella Matrices and Higher Curvatures of Curve"*. In this section, an (nxn)-type umbrella matrix is obtained using the curvature matrix of the curve-hypersurface couple in the Cayley Formula, and the relationship between the Darboux matrix of the umbrella motion and the curvature matrix is given. Then, an infinitesimal umbrella motion is obtained using the umbrella matrix. In addition, Yusuf Yaylı's also contribution to umbrella matrices is in his doctoral thesis. In this thesis, quaternions are discussed. Hamilton motions and Lie groups were analyzed. Umbrella matrices were obtained from Hamilton's motions with the help of the Cayley Formula.

2. Ertuğrul Özdamar's Contributions to The Subject

Firstly, let us start by defining the umbrella matrix.

Definition 2.1 The orthogonal matrix $A \in O(n, \mathbb{C})$ such that $AS = S$ is called an umbrella matrix, where $S = \begin{bmatrix} 1 & 1 \dots & 1 \end{bmatrix}^T \in \mathbb{R}_1^n$ and the set of umbrella matrices $A(n, \mathbb{C})$.

This definition above is of course similar for $\mathbb{R}$ and we may called $A(n)$ is the set of umbrella matrices. So we are talking about an orthogonal rotation matrix, which is a matrix that keeps the $S = \begin{bmatrix} 1 & 1 \dots & 1 \end{bmatrix}^T$ axis fixed and rotates everything around this axis. We may give some situations these matrices have brought to the following [2].

Theorem 2.2 Let $A(n, \mathbb{C})$ is the set of umbrella matrices, then

- $A(n, \mathbb{C}) \subseteq O(n, C)$ is subgroup.
- $A(n) \subseteq SO(n)$ is subgroup.
- $A(n)$ and $A(n, \mathbb{C})$ are the Lie group.
- The $A(n)$ Lie group is compact and connected [2].

We can also talk about the Lie algebra of this group. The umbrella Lie group $A(n)$ has a Lie algebraic structure, and let us denote it by $\chi(A(n))$. Let the lie algebra of the $O(n-1)$ orthogonal group be $\chi(O(n-1))$. It is known that $gl(n-1, \mathbb{R})$ is isomorphic to the space of skew-symmetric matrices $\chi(O(n-1))$. Ertugrul Özdamar showed that there is a Lie algebra isomorphism between $\chi(O(n-1))$ and $\chi(A(n))$.

Theorem 2.3 There is a Lie algebra isomorphism between the Lie algebras $\chi(A(n))$ and $\chi(O(n-1))$.

Therefore, A Lie algebra isomorphism can be established between $gl(n-1, \mathbb{R})$ and $\chi(A(n))$.

Now let us consider group $A(n-1)$ as a subgroup in $A(n)$. Hence, for each $A \in A(n-1)$

$$\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \in A(n)$$

Moreover, for the matrix

$$S = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix}$$

we may give the σ transformation following

$$\sigma : A(n) \longrightarrow A(n)$$

This σ be defined as $\sigma(A) = SAS^{-1}$. Thus, the following theorem can be mentioned [2].

Theorem 2.4 $(A(n), A(n-1), \sigma)$ is a symmetric space [2].

Definition 2.5 The symmetric space $(A(n), A(n-1), \sigma)$ is called Umbrella symmetric space [2].

Definition 2.6 The symmetric Lie algebra that the Umbrella symmetric space denotes is called Umbrella symmetric Lie algebra [2].

3. Nuri Kuruoğlu's Contributions to The Subject

Let us start by giving the umbrella matrix definition in this section in a different way from def.().

Theorem 3.1 A matrix $A \in GL(n, \mathbb{R})$ is Umbrella Matrix, if

$$AS = S$$

where $S = \begin{bmatrix} 1 & 1 \dots & 1 \end{bmatrix}^T \in \mathbb{R}_1^n$ and $[\ ]^T$ denotes the transpose of a matrix and $GL(n, R)$ is the set of all $n \times n$, nonsingular matrices. The set of Umbrella Matrices will be denoted by H(n) [3].

In this section, some crucial theorems can be given for $H(n)$, which is defined as the set of umbrella matrices.

Theorem 3.2 Let $A(n) \subseteq SO(n)$ is the set of umbrella matrices, then

- $A(n) \subseteq H(n)$ is subgroup.
- $A(n)$ is Lie group of $H(n)$ [3].

Next, we will give a definition that brings a different approach to umbrella matrices from Nuri Kuruoğlu's doctoral thesis.

Definition 3.3 A matrix $A \in GL(n, \mathbb{R})$ is a Double Umbrella matrix if

$$AS = S$$

$$A^T S = S$$

where $S = \begin{bmatrix} 1 & 1 \dots & 1 \end{bmatrix}^T \in \mathbb{R}_1^n$, $[\ ]^T$ denotes the transpose of matrix and $GL(n, \mathbb{R})$ is the set of all $n \times n$, nonsingular matrices. The set Double Umbrella matrices is denoted by $DU(n)$. [4]

Theorem 3.4 Let $DU(n)$ is the set of Double Umbrella matrices, then

- $A(n) \subseteq DU(n)$ is subgroup.
- $DU(n) \subseteq H(n)$ is subgroup.
- $DU(n)$ is Lie group of $H(n)$.
- $DU(n)$ is a Lie subgroup of $GL(n, \mathbb{R})$[3].

4. Erdoğan Esin's Contributions to The Subject

In this part, we will discuss Erdogan Esin's paper titled *"Umbrella Matrices And Higher Curvature Of A Curve"*, which reviews his doctoral thesis. While doing this review, we will also talk about some basic information. Firstly, let us start by giving the curvature matrix of the curve on a hypersurface.

Let M be a hypersurface in $\mathbb{E}^n$ and α curve on M. The derivative formulas of the natural frame field $\{X_1, \dots, X_n\}$ are

$$\begin{aligned} D_{X_1}X_i &= X_i^{'} = -k_{(i-1)g}X_{i-1} + k_{ig}X_{i+1} + II(X_1, X_i)X_n \\ D_{X_1}X_n &= -II(X_1, X_1)X_1 - II(X_1, X_2)X_2 - \ldots - II(X_1, X_{n-1})X_n \end{aligned}$$

where $1 \leq i \leq n-1$ and $k_{0g} = k_{(n-1)g} = 0$. In the matrix form, these derivative formulas become

$$\begin{bmatrix} X_1^{'} \\ X_2^{'} \\ \vdots \\ X_{n-1}^{'} \\ X_n^{'} \end{bmatrix} = \begin{bmatrix} 0 & k_{1g} & 0 & \ldots & 0 & 0 & II(X_1, X_1) \\ -k_{1g} & 0 & k_{2g} & \ldots & 0 & 0 & II(X_1, X_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -k_{(n-2)g} & 0 & II(X_1, X_{n-1}) \\ -II(X_1, X_1) & \ldots & \ldots & \ldots & \ldots & -II(X_1, X_{n-1}) & 0 \end{bmatrix}$$

or simply

$$X^{'} = K(x)X$$

The matrix $K(x)$ in called the (higher) curvature matrix of the pair (α, M) [6].

We assume that the directions $X_2, X_3, \ldots, X_{n-2}$ of the natural frame field $X = \{X_1, \ldots, X_n\}$ are conjugate directions with the tanget direction X_1 for a curve α which is different from the line of curvature on a hypersurface M in E^n. Then the higher curvature matrix can be written in the form

$$K(X) = \begin{bmatrix} 0 & k_{1g} & 0 & \ldots & 0 & 0 & II(X_1, X_1) \\ -k_{1g} & 0 & k_{2g} & \ldots & 0 & 0 & 0 \\ 0 & -k_{2g} & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -k_{(n-2)g} & 0 & II(X_1, X_{n-1}) \\ -II(X_1, X_1) & \ldots & \ldots & \ldots & \ldots & -II(X_1, X_{n-1}) & 0 \end{bmatrix} \quad (1)$$

since

$$II(X_1, X_2) = II(X_1, X_3) = \ldots = II(X_1, X_{n-2}) = 0.$$

Let us write

$$\begin{aligned} k_{ig} &= b_{i+2} \qquad (1 \leq i \leq n-2) \\ II(X_1, X_{n-1}) &= b_1 \\ II(X_1, X_1) &= b_2. \end{aligned}$$

Thus the equality (1), with respect to the elements b_j $(1 \leq j \leq n)$, takes the form

$$K(X) = \begin{bmatrix} 0 & b_3 & 0 & \dots & 0 & 0 & -b_2 \\ -b_3 & 0 & b_4 & \dots & 0 & 0 & 0 \\ 0 & -b_4 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -b_n & 0 & b_1 \\ b_2 & 0 & 0 & \dots & 0 & -b_1 & 0 \end{bmatrix} \tag{2}$$

Let A be any $n \times n$ orthogonal matrix one of the characteristic values of which is not -1. Then A can always be expressed as

$$A = (I_n - B)^{-1}(I_n + B) \tag{3}$$

where B is as $n \times n$ skew-symmetric matrix. The formula (3) is known as the Cayley formula. Now taking $K(X)$ instead of B in this formula we obtain the following theorem [1].

Theorem 4.1 For $n \geq 3$, if $b_1 = b_2 = \ldots = b_n = c$ then the orthogonal matrix

$$A = (I_n - K(X))^{-1}(I_n + K(X)) \tag{4}$$

one of the charaeteristic values of which is not -1, is an umbrella matrix [1].

Next, we will give examples of surfaces formed by a (3×3)-type curvature matrix and an orbital curve.

Example 4.2 The curvature matrix of an α curve in 3-dimensional Euclidean space $\mathbb{E}^3$ has the form

$$K(x) = \begin{pmatrix} 0 & \kappa_g & -\kappa_n \\ -\kappa_g & 0 & \tau_g \\ \kappa_n & -\tau_g & 0 \end{pmatrix}$$

If we take $\kappa_g = \kappa_n = \tau_g = u$, we get

$$K(x) = \begin{pmatrix} 0 & u & -u \\ -u & 0 & u \\ u & -u & 0 \end{pmatrix}$$

and using Cayley's formula, the orthogonal matrix

$$A = \frac{1}{1+u^2} \begin{pmatrix} 1-u^2 & 2(u^2+u) & 2(u^2-u) \\ 2(u^2-u) & 1-u^2 & 2(u^2+u) \\ 2(u^2+u) & 2(u^2-u) & 1-u^2 \end{pmatrix}$$

is an umbrella matrix [1].

Let we consider A umbrella matrix with the orbit curve $\alpha(v) = (sinv, v, 0)$, then we obtain the following surface

$$H(u,v) = (\frac{(1-u^2)sinv + (2u^2+2u)v)}{1+3u^2}, \frac{(2u^2-2u)sinv + (1-u^2)v)}{1+3u^2}, \frac{(2u^2+2u)sinv + (2u^2-2u)}{1+3u^2}$$

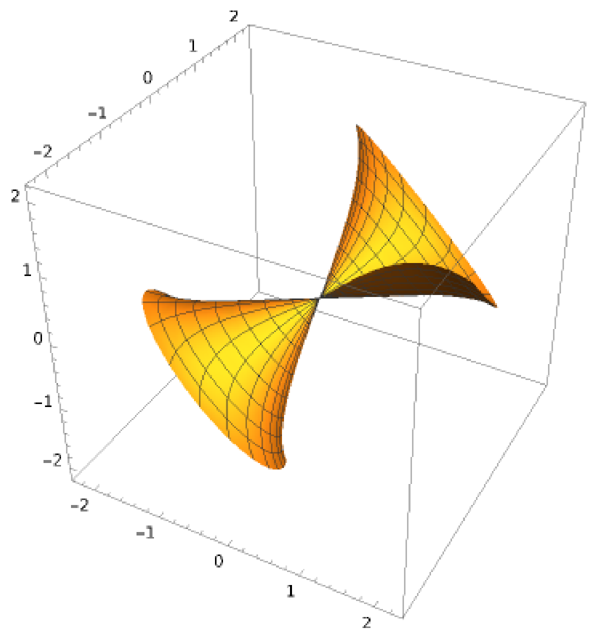

FİG. 1. H Umbrella Surface

The picture of the surface of H is rendered in Figure 1. In addition, considering the orbital curve $\beta(v) = (v, v^3, 0)$, we obtain the following surface

$$G(u,v) = (\frac{(1-u^2)v + (2u^2+2u)v^3)}{1+3u^2}, \frac{(2u^2-2u)v + (1-u^2)v^3)}{1+3u^2}, \frac{(2u^2+2u)v + (2u^2-2u)v^3)}{1+3u^2}).$$

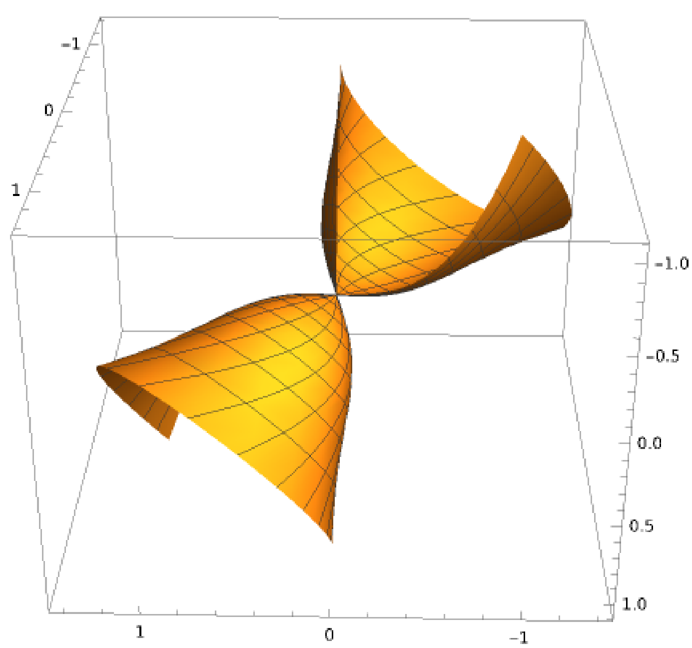

FİG. 2. G Umbrella Surface

The picture of the surface of H is rendered in Figure.

Now we may give a relation between the Darboux matrix of the umbrella motion and the higher curvature matrix by the following theorem.
Theorem 4.3 Let $W(A)$ be the Darboux matrix of the umbrella motion, where A is given by (4), and $K(X)$ be the higher curvature matrix. Then

$$W(A) = \frac{2c^{'}}{c}(I_n - K(X))^{-1}K(X)(I_n + K(X))^{-1}$$

where $c = c(s)$ [1].
We will talk about the definition of infinitesimal transformation. An infinitesimal linear transformation is defined as a transformation whose matrix is

$$A = I_n + \varepsilon[b_{ij}]$$

where $[b_{ij}]$ skew-symmetric matrix and ε denotes an infinitesimal quantity of the first order [7]. Thus, we obtain the following theorem.
Theorem 4.4 Let $W(A)$ be the Darboux matrix of the umbrella motion, where A is given by (4). Then the matrix $I_n + W(A)ds$ is also an infinitesimal umbrella matrix [1].

Conclusion

As mentioned in all these studies, the kinematic applications of umbrella matrices come to the fore. Here, some umbrella matrices were obtained from some specially selected skew-symmetric matrices with the help of Cayley's Formula. We will answer the question of how skew-symmetric matrices will be in such a way as to obtain all umbrella matrices with a study in the future.

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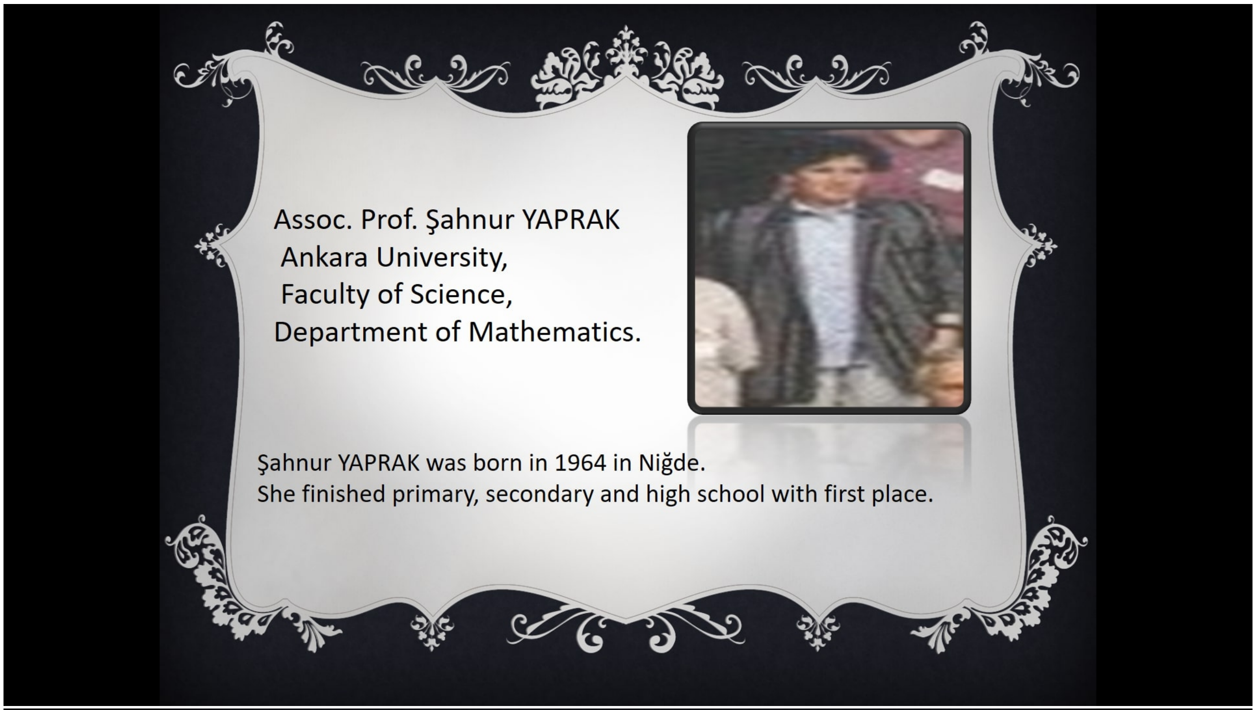
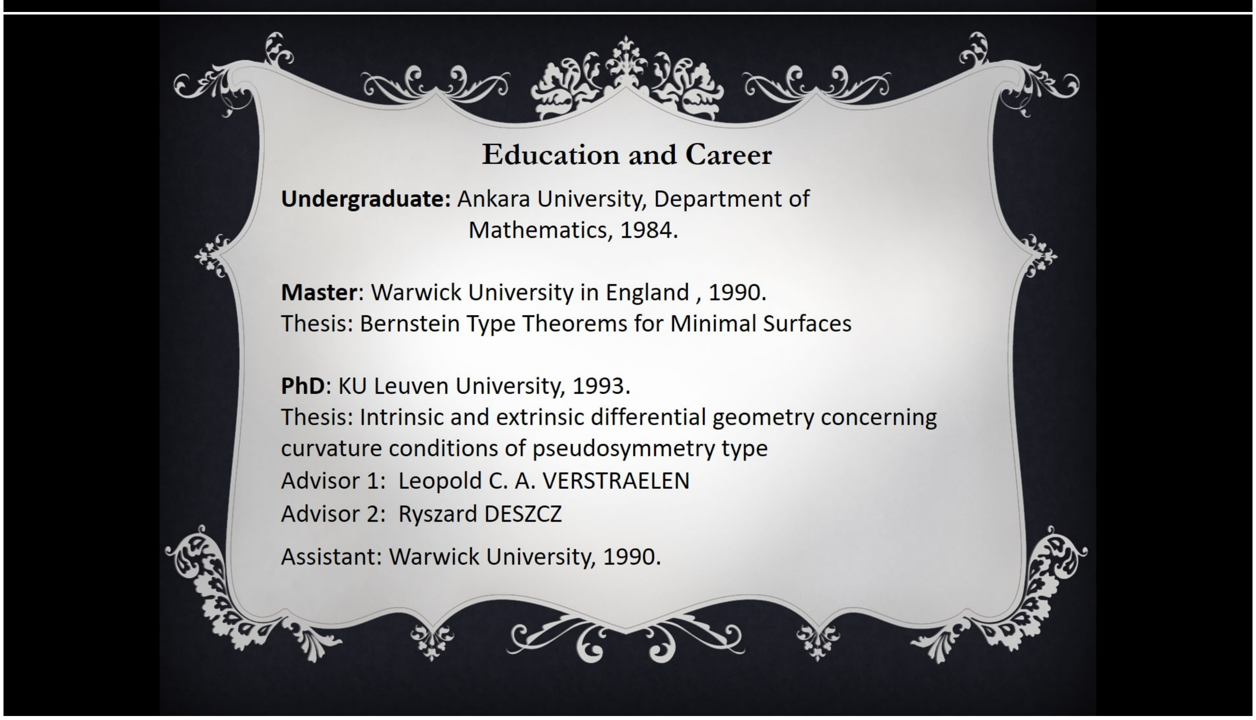
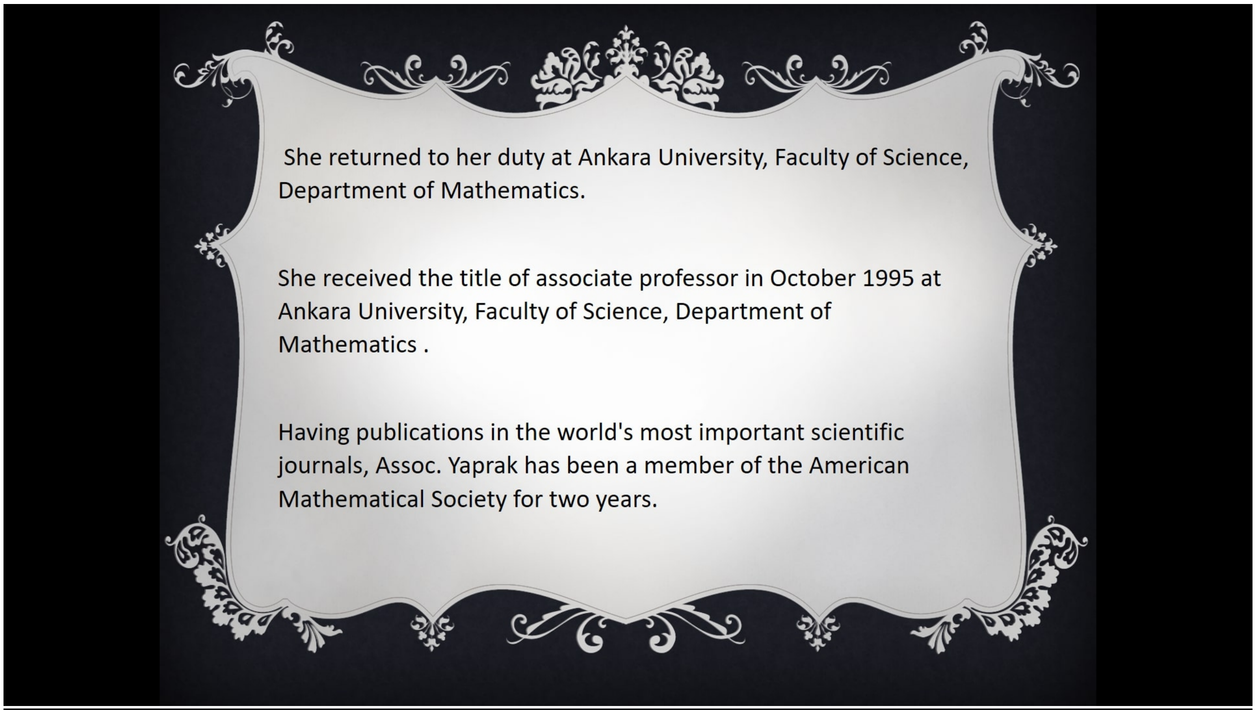
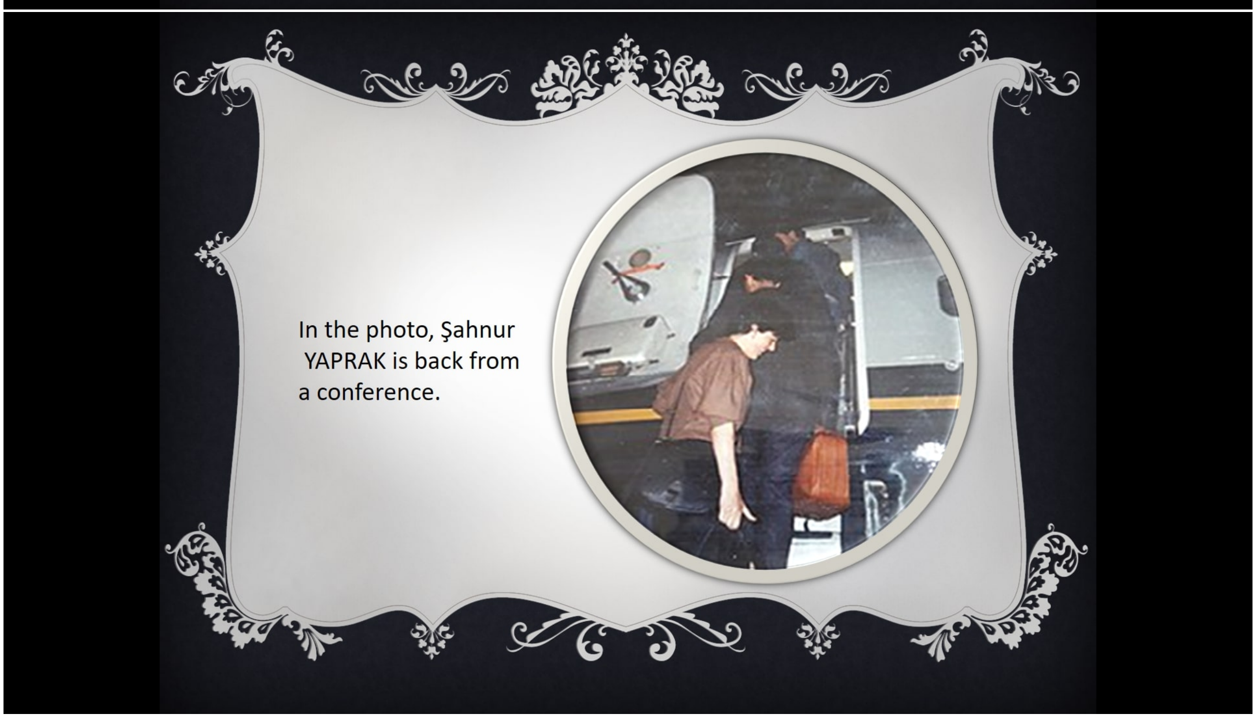
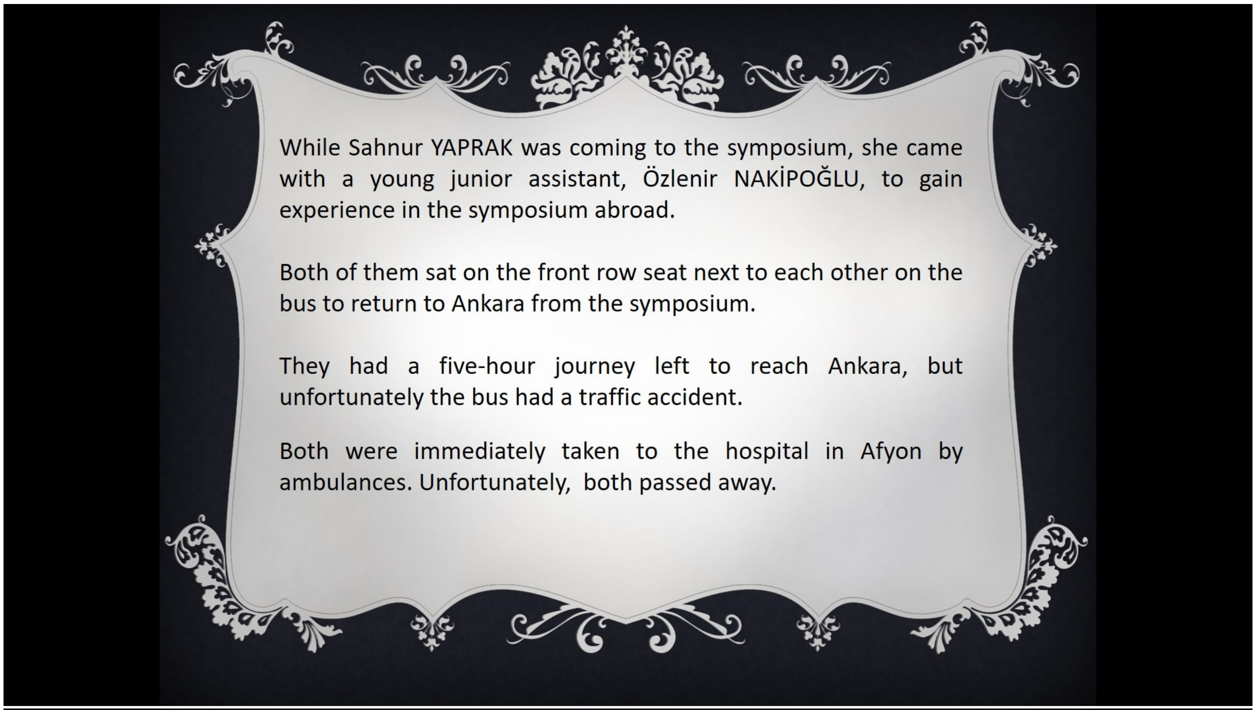
Şahnur YAPRAK's Life

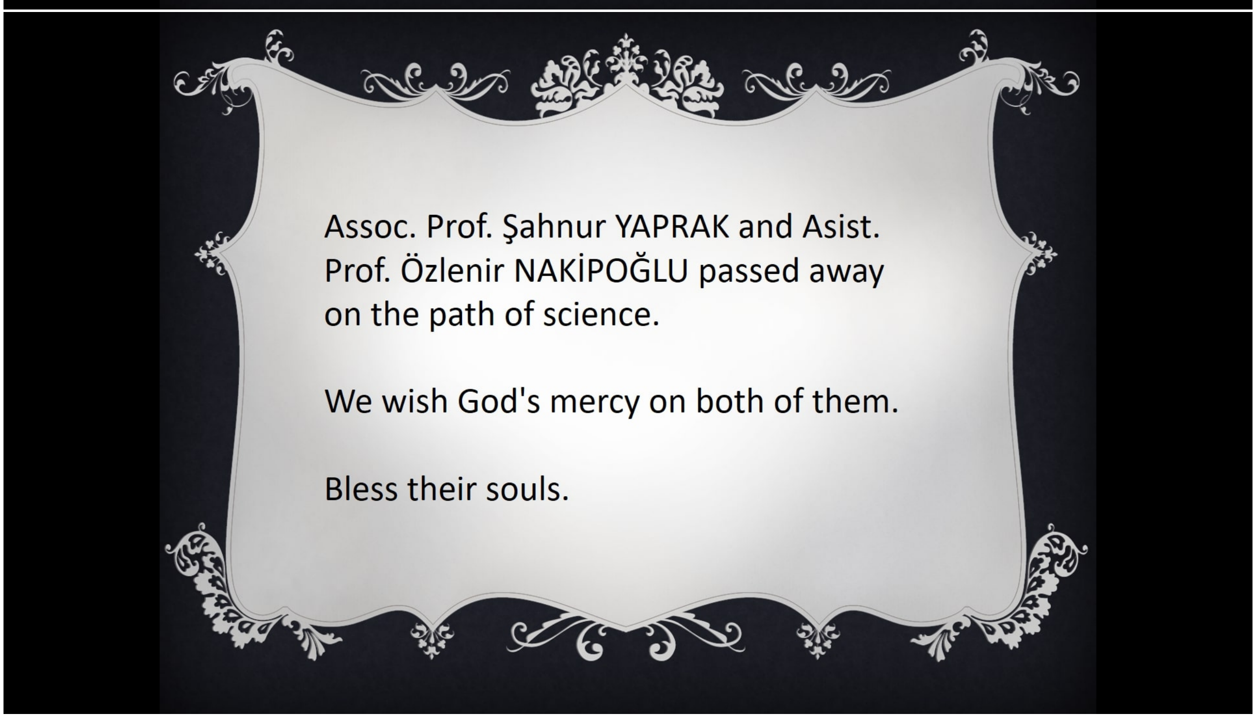


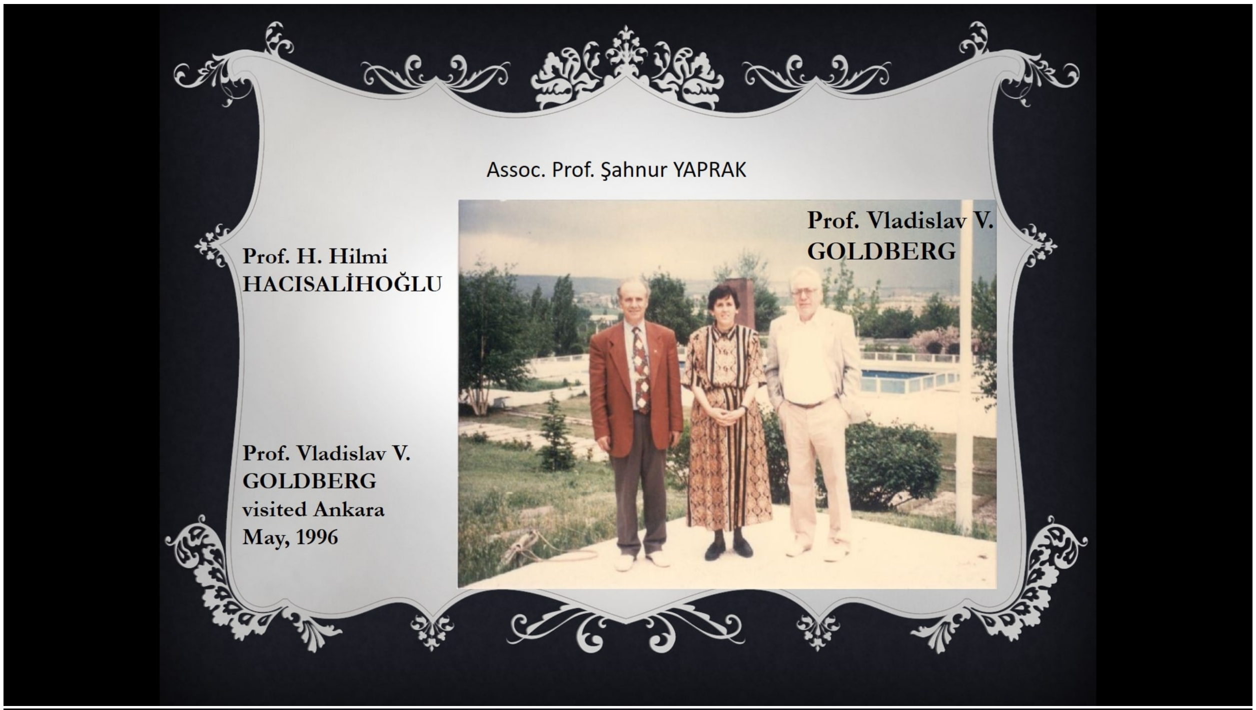


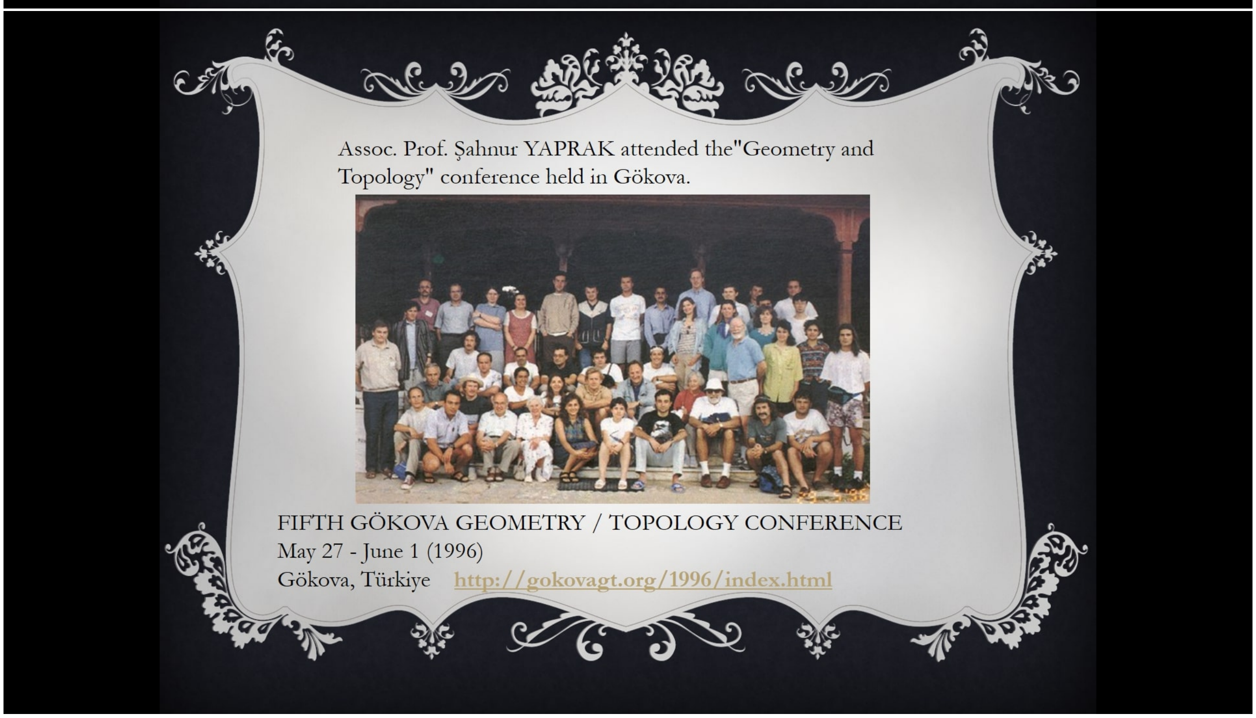

She returned to her duty at Ankara University, Faculty of Science, Department of Mathematics.
She received the title of associate professor in October 1995 at Ankara University, Faculty of Science, Department of Mathematics .
Having publications in the world's most important scientific journals, Assoc. Yaprak has been a member of the American Mathematical Society for two years.

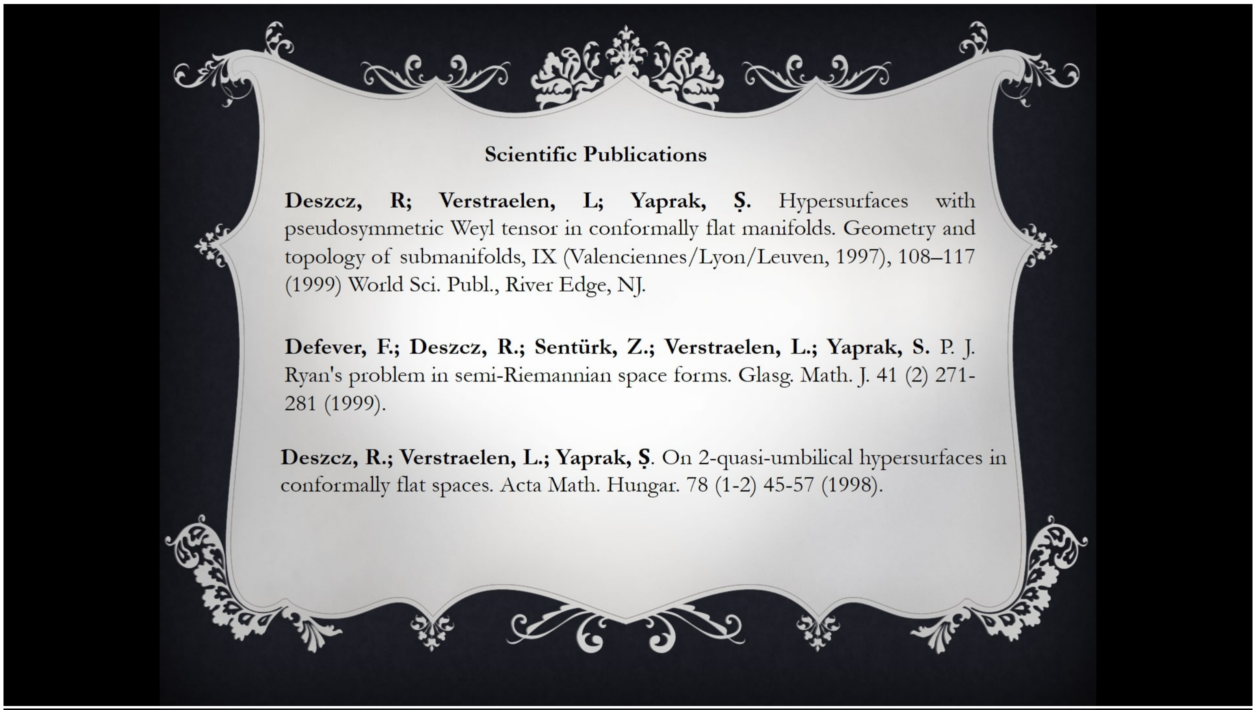

In the photo, Şahnur YAPRAK is back from a conference.


While Sahnur YAPRAK was coming to the symposium, she came with a young junior assistant, Özlenir NAKİPOĞLU, to gain experience in the symposium abroad.
Both of them sat on the front row seat next to each other on the bus to return to Ankara from the symposium.
They had a five-hour journey left to reach Ankara, but unfortunately the bus had a traffic accident.
Both were immediately taken to the hospital in Afyon by ambulances. Unfortunately, both passed away.


Assoc. Prof. Şahnur YAPRAK and Asist. Prof. Özlenir NAKİPOĞLU passed away on the path of science.
We wish God's mercy on both of them.
Bless their souls.


Assoc. Prof. Şahnur YAPRAK
Prof. Vladislav V. GOLDBERG
Prof. H. Hilmi HACISALİHOĞLU
Prof. Vladislav V. GOLDBERG visited Ankara May, 1996


Assoc. Prof. Şahnur YAPRAK attended the"Geometry and Topology" conference held in Gökova.
FIFTH GÖKOVA GEOMETRY / TOPOLOGY CONFERENCE
May 27 - June 1 (1996)
Gökova, Türkiye http://gokovagt.org/1996/index.html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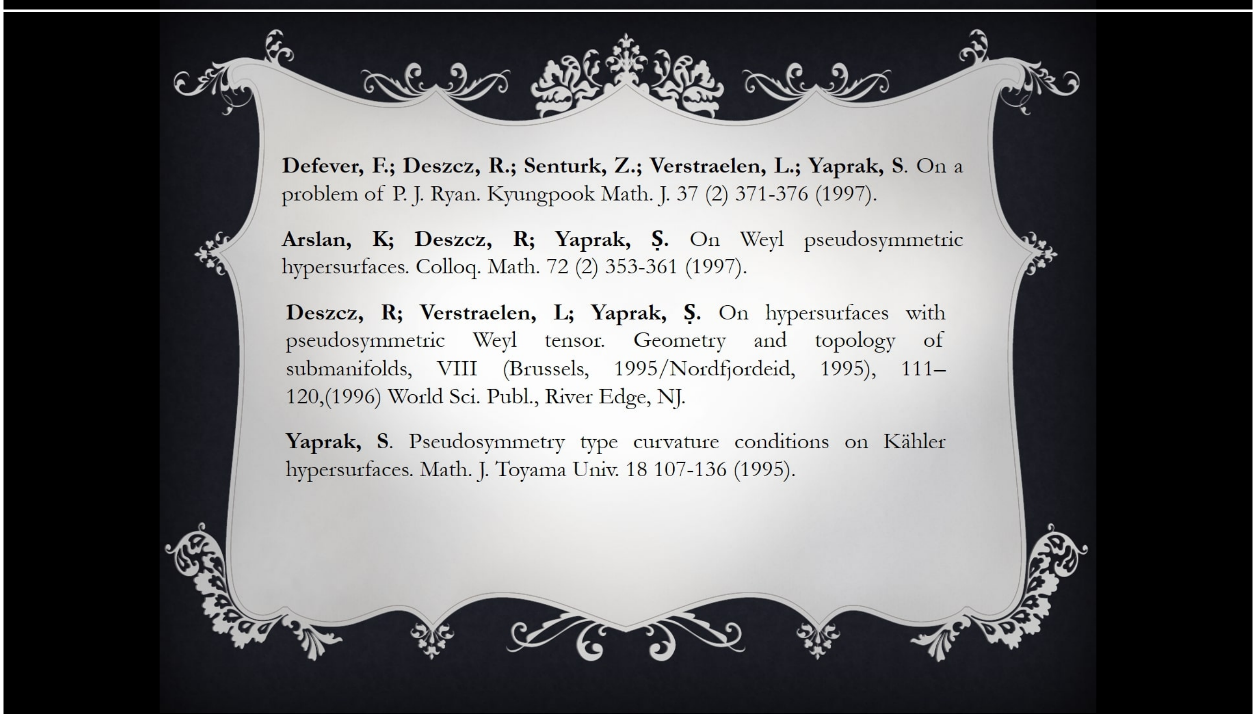
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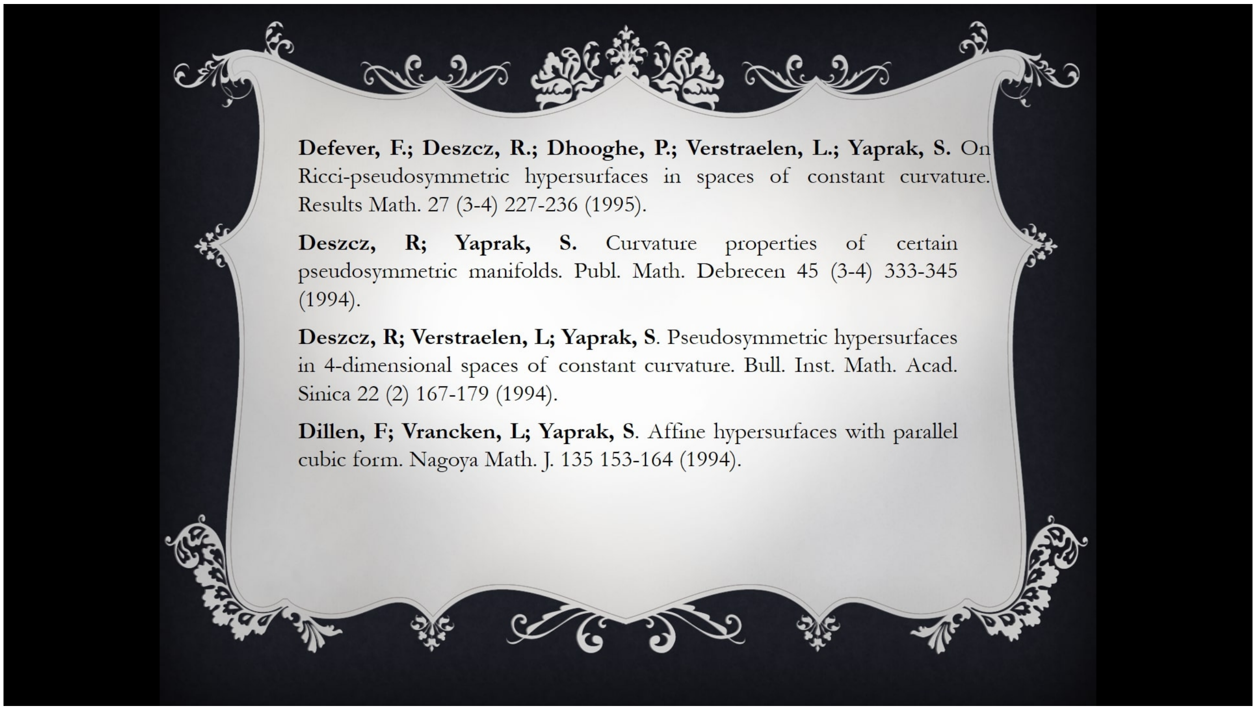
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Dear invited speakers and participants

It was actually a very melancholic symposium, I'm sorry about that. There are many more scientists who died on the path of science I commemorate all of them with mercy I hope that the symposium was good and useful. This symposium was the first international symposium that I organized. If we made a mistake unknowingly, we seek your forgiveness.

I would like to thank the invited speakers and our young friends who made presentations. I would like to thank the symposium scientific committee and the chairperson of the sessions. I think that the symposium was very beneficial for our young colleagues who participated as listeners and we hope that they learned a lot of information. Scientists who talk in symposiums must have an audience so that what was told is useful. Thank you to everyone who participated as a listener.

I would also like to thank the organizing committee for helping me organize the symposium.

Wishing you all a healthy and happy day.

I greet you all with my deepest feelings.

Organizing Committee Chairperson
Prof. Aysel TURGUT VANLI

Participant List

Sudhakar Kumar CHAUBEY, University of Tech. and Appl. Sci., Shinas (Oman), Invited Speaker

Ryszard DESZCZ, Wrocław University of Environmental and Life Sciences (Poland), Invited Speaker

Sorin DRAGOMIR, Università degli Studi della Basilicata (Italy), Invited Speaker

Dorel FETCU, Asachi Technical University of Iasi (Romania), Invited Speaker

Eduardo GARCIA-RIO, University of Santiago de Compostela (Spain), Invited Speaker

Mohammad Nazrul Islam KHAN, Qassim University (Saudi Arabia), Invited Speaker

Adela MIHAI, Technical University of Civil Engineering Bucharest (Romania), Invited Speaker

Sema SALUR, University of Rochester (USA), Invited Speaker

Venkatesha VENKATESHA, Karnata University (India), Invited Speaker

Chris WOOD, University of York (UK), Invited Speaker

Şeyma AFŞAR, Gazi University (Türkiye), Listener

Alev KELLECİ AKBAY, Iskenderun Technical University (Türkiye), Listener

Ömer AKGÜLLER, Muğla Sıtkı Koçman University (Türkiye), Listener

Çisem Güneş AKTAŞ, Abdullah Gül University (Türkiye), Oral Presentation

Gamze ALKAYA, Gazi University (Türkiye), Listener

Anıl ALTINKAYA, Gazi University (Türkiye), Listener

Rümeysa Sacide ALTINKAYA, Gazi University (Türkiye), Listener

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Abir BOUNAAMA, 20 August 1955 (Algeria), Oral Presentation

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FACULTY OF SCIENCE
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In Memory of
Prof. Erdoğan ESİN