Graham-Lee connection[1] and the C^∞ regularity up to the boundary problem on pseudoconvex domains

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Let $K(\zeta, z)$ be the Bergman kernel of a smoothly bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$, $n \geq 2$, and let g_B and Δ_B be respectively the Bergman metric on Ω

$$(1) \qquad \left(g_B\right)_{jk} = \frac{\partial^2 \log K(z, z)}{\partial z^j \, \partial \overline{z}^k}$$

and the Laplace-Beltrami operator of the Kählerian manifold (Ω, g_B). C.R. Graham & J.M. Lee considered (cf. [GrLe]) the Dirichlet problem

$$(2) \qquad \Delta_B u = 0 \quad \text{in } \Omega, \quad u = f \quad \text{on } \partial\Omega,$$

with $f \in C^\infty(\partial\Omega)$ and studied the corresponding C^∞ regularity up to the boundary problem, a first step of which is to look for the compatibility equations $\mathscr{C}(f) = 0$ along $\partial\Omega$ that the boundary values of a solution $u \in C^\infty\left(\overline{\Omega}\right)$ to the Dirichlet problem (2) must satisfy. One purpose of the present talk is to explain the geometric analysis ingredients in C.R. Graham & J.M. Lee's approach, relying on the construction of a canonical linear connection ∇ (the *Graham-Lee connection*) on a one-sided neighborhood V of the boundary $\partial\Omega$, foliated by level sets of the defining function

$$(3) \qquad \varphi(z) = -K(z, z)^{-1/(n+1)}$$

of Ω, whose pointwise restriction to a leaf $\{\varphi = -\delta\}$, $\delta > 0$, is the Tanaka-Webster connection of that leaf. Once the defining function (3) is fixed, the real differential 1-form $\theta \in \Omega^1(\partial\Omega)$ given by

$$(4) \qquad \theta = \mathbf{j}^* \left\{ \frac{i}{2} \left(\overline{\partial} - \partial\right) \varphi \right\}, \quad \mathbf{j} : \partial\Omega \hookrightarrow \mathbb{C}^n,$$

is a contact form on $\partial\Omega$ and the formulas (1) and (4) exhibit an explicit relationship between the Kählerian geometry of the interior of the domain Ω and the contact geometry of its boundary. The very choice (3)

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springs from Fefferman's asymptotic formula (cf. [Fef1])

$$K(z,\, z) = C_\Omega \left|\nabla\varphi(z)\right|^2 \cdot \det\, L_\varphi(z) \cdot |\varphi(z)|^{-(n+1)} + H(z,\, z),$$
$$|H(z,z)| \leq C'_\Omega |\varphi(z)|^{-(n+1)+1/2} \cdot \left|\log\, |\varphi(z)|\right|.$$

The great protagonists of the present talk, the giants to whom the line of thought described in the present talk is due, are without any doubt the scientists Charles Fefferman, Robin C. Graham, and John M. Lee. We also report on other applications of the Graham-Lee connection, to the Dirichlet problem for the Bergman-harmonic map system (cf. [DrKa]) and for the Yang-Mills equation (cf. [BDU]). We discuss the possibility of extending C.R. Graham & J.M. Lee's geometric analysis to Dirichlet problems (2) over domains whose boundaries have degenerate Levi forms e.g. Diederich-Fornaess "worm" domains $\Omega \subset \mathbb{C}^2$.

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