ON ROTER MANIFOLDS

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ABSTRACT

In this talk we present results on Roter manifolds and their generalizations.

Let g, R, S, S^2, κ and C be the metric tensor, the Riemann-Christoffel curvature tensor, the Ricci tensor and its square, the scalar curvature and the Weyl conformal curvature tensor of a semi-Riemannian manifold (M, g), $\dim M = n \geqslant 4$, respectively. Moreover, let (M, g) be a non-Einstein and non-conformally flat manifold and let $\mathcal{U}$ be the set of all points of M at which the tensor S is not proportional to the tensor g and C is a non-zero tensor. We will assume that $\mathcal{U}$ is a non-empty set.

The tensor R (or, equivalently, the tensor C) of some 2-recurrent spaces, essentially conformally symmetric manifolds, as well as pseudosymmetric manifolds (and in particular, pseudosymmetric hypersurfaces in spaces of constant curvature) is a linear combination of the Kulkarni-Nomizu products: $g \wedge g$, $g \wedge S$ and $S \wedge S$. Manifolds with such tensor R are called Roter manifolds or Roter spaces. Precisely, the manifold (M, g), $n \geqslant 4$, is said to be a Roter manifold or a Roter space, if at every point of $\mathcal{U} \subset M$ the tensor R satisfies the equation

$$R = \frac{\phi}{2}\, S \wedge S + \mu\, g \wedge S + \frac{\eta}{2}\, g \wedge g, \tag{1}$$

where ϕ, μ, η are some functions on $\mathcal{U}$. It is easy to check that at every point of $\mathcal{U}$ the tensor S^2 is a linear combination of the tensors g and S. Further, using (1) we can prove that

$$C = \frac{\phi}{n-2}\left(g \wedge S^2 + \frac{n-2}{2}\, S \wedge S - \kappa\, g \wedge S + \frac{(\kappa)^2 - \mathrm{tr}_g(S^2)}{2(n-1)}\, g \wedge g \right) \tag{2}$$

on $\mathcal{U}$. Roter manifolds (M, g), $n \geqslant 4$, satisfy on $\mathcal{U} \subset M$ several pseudosymmetry type curvature conditions. In particular, the $(0, 6)$ tensors: $C \cdot R$, $R \cdot C$, $Q(S, C)$ and $Q(g, C)$ satisfy on $\mathcal{U}$

$$C \cdot R - R \cdot C = Q(S, C) - \frac{\kappa}{n-1}\, Q(g, C). \tag{3}$$

Evidently, this condition holds at all points at which the tensor C vanishes. It is also satisfied on every semi-Riemannian Einstein manifold of dimension $\geqslant 4$. Thus (3) is satisfied on every Roter manifold. We mention that (3) is an example of a generalized Einstein metric condition.

In the class of warped product manifolds $\overline{M} \times_F \mathbb{S}^{n-2}(1)$, with a 2-dimensional base manifold $(\overline{M}, \overline{g})$, a warping function F and an $(n-2)$-dimensional standard unit sphere $\mathbb{S}^{n-2}(1)$, $n \geqslant 4$, and

the line element

$$ds^2 = -h(r)\, dr^2 + \frac{1}{h(r)}\, dr^2 + r^2\, d\Omega_{n-2}^2,$$

where $h = h(r)$ is a positive smooth function on $\overline{M}$ and $d\Omega_{n-2}^2$ is the line element of $\mathbb{S}^{n-2}(1)$, there are also Roter spacetimes. In particular, the Reissner-Nordström, the Reissner-Nordström-de Sitter and the Reissner-Nordström-anti-de Sitter spacetimes are Roter spacetimes.

Non-Einstein and non-conformally flat hypersurfaces M, $\dim M = n \geqslant 4$, in an $(n+1)$-dimensional space of constant curvature having at every point of $\mathcal{U} \subset M$ exactly two distinct principal curvatures are Roter hypersurfaces. Thus in particular, non-Einstein and non-conformally flat Clifford hypersurfaces, of dimension $\geqslant 4$, are Roter hypersurfaces.

Some Roter manifolds admitting geodesic mappings.

Study on hypersurfaces in space of constant curvature with exactly three distinct principal curvatures, as well as on 2-quasi Einstein warped product manifolds, lead to an extension of the class of the Roter manifolds. Let (M, g), $n \geqslant 4$, be a non-Einstein and non-conformally flat manifold. The manifold (M, g) is called a generalized Roter manifold, or a generalized Roter space, if at every point of $\mathcal{U} \subset M$ the tensor R (or, equivalently, the tensor C) is a linear combination of the Kulkarni-Nomizu products formed by the tensors: $g, S, S^2, \ldots, S^p$, where p is some natural number ≥ 2. In particular, if $p = 2$ then the tensor R of a generalized Roter manifold satisfies on $\mathcal{U}$

$$R = \frac{\phi_3}{2}\, S^2 \wedge S^2 + \phi_2\, S \wedge S^2 + \frac{\phi_1}{2}\, S \wedge S + \mu_2\, g \wedge S^2 + \mu_1\, g \wedge S + \frac{\eta_1}{2}\, g \wedge g, \tag{4}$$

where $\phi_1, \phi_2, \phi_3, \mu_1, \mu_2, \eta_1$ are some functions on $\mathcal{U}$. Clearly, (2) is a special form of (4).

Let $\overline{M} \times_F \widetilde{N}$ be the warped product manifold, with 2-dimensional base manifold $(\overline{M}, \overline{g})$, a warping function F, and $(n-2)$-dimensional fiber $(\widetilde{N}, \widetilde{g})$, $n \geqslant 4$, and let $(\widetilde{N}, \widetilde{g})$ be a semi-Riemannian space, assumed to be of constant curvature when $n \geqslant 5$. In the class of these warped product manifolds $\overline{M} \times_F \widetilde{N}$ there are also generalized Roter manifolds satisfying (2) which are not Roter manifolds. Namely, certain spacetimes of the form $\overline{M} \times_F \mathbb{S}^2(1)$, $\dim \overline{M} = 2$, non-Roter manifolds, satisfy (2). For instance, in the class of general static spherically symmetric wormholes, i.e., spacetimes with the spherically symmetric static Morris-Thorne wormhole metric

$$ds^2 = -\exp(2\psi(r))\, dr^2 + \left(1 - \frac{b(r)}{r}\right)^{-1} dr^2 + r^2\, d\Omega_2^2,$$

where $b = b(r)$ and $\psi = \psi(r)$ are identified as the shape and redshift functions, respectively, there are also generalized Roter manifolds.

Warped product manifolds $\overline{M} \times_F \widetilde{N}$, with a 2-dimensional Riemannian manifold $(\overline{M}, \overline{g})$, a warping function F and an $(n-2)$-dimensional sphere $\mathbb{S}^{n-2}(1)$, $n \geqslant 4$, are related to Chen ideal submanifolds. Namely, some Chen ideal submanifolds M of dimension n in the Euclidean space $\mathbb{E}^{n+m}$, $n \geqslant 4$, $m \geqslant 1$, are isometric to an open submanifold of a warped product manifold $\overline{M} \times_F \mathbb{S}^{n-2}(1)$, of a 2-dimensional base manifold $(\overline{M}, \overline{g})$ and the sphere $\mathbb{S}^{n-2}(1)$, where the warping function F is a solution of some second order quasilinear elliptic partial differential equation in the plane. Condition (2) is satisfied on the set $\mathcal{U}$ of such submanifolds.

Keywords warped product manifold · spacetime · hypersurface · Chen ideal submanifold · pseudosymmetry type curvature condition · Roter manifold · generalized Roter manifold

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