

Lecture 1

CR Manifolds, Tanaka-Webster Connection, Tangential Cauchy-Riemann Complex, Kohn-Rossi Cohomology, Subelliptic Theory¹

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Dirichlet problem for the Cauchy-Riemann equations on the Siegel domain $\Omega = \{(z, \zeta) \in \mathbb{C}^2 : \frac{1}{2}(\zeta + \bar{\zeta}) > z\bar{z}\}$, tangential C-R equations $\bar{\partial}_b f = 0$, Hans Lewy's legacy (cf. [Lew]). Cauchy-Riemann (CR) structures $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$ on a $(2n+1)$ -dimensional C^∞ manifold M . The unsolvable Lewy operator and Denson Clyde Hill's nonembeddable examples (cf. [Hil]). The maximal complex distribution $H(M) = \text{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}$ in the classical work by Beniamino Segre, Tulio Levi-Civita, and Eugenio Elia Levi (cf. [Seg], [LeCi], and [Lev]), the conormal (real line) bundle $H(M)^\perp \subset T^*(M)$ over an orientable CR manifold, Sydney M. Webster's pseudohermitian structures $\theta \in C^\infty(H(M)^\perp)$. The Levi form, nondegenerate (respectively positive definite) CR structures and (positively oriented) contact forms, the Reeb vector field, the Tanaka-Webster connection ∇ of a nondegenerate CR manifold M , equipped with a contact form θ (cf. [Tan], [Web], and [DrTo], pp. 5-49). The tangential Cauchy-Riemann complex $\bar{\partial}_b^{(q)} : \Omega^{0,q}(M) \rightarrow \Omega^{0,q+1}(M)$, $q \geq 0$, and the Kohn-Rossi cohomology $H^{0,q}(M) = H^q(\Omega^{0,\bullet}(M), \bar{\partial}_b^{(\bullet)})$. CR-holomorphic vector bundles $(E, \bar{\partial}_E) \rightarrow M$, Tanaka connection, Frölicher spectral sequence (cf. [Tan]). Subelliptic estimates, Kohn-Hodge-de Rham theory on compact strictly pseudoconvex CR manifolds (cf. [Koh] and [BDSS], pp. 237-294). CR manifolds with conical singularities, CR orbifolds (cf. [Sat], [GiNi], [DrMa]).

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