Lecture 3

Pseudohermitian Geometry

versus

Lorentzian Geometry: Fefferman's Metric[1]

Sorin Dragomir[2]

Canonical complex line bundle $\mathbb{C} \to K(M) \to M$ and canonical circle bundle $S^1 \to C(M) \xrightarrow{\pi} M$ over a CR manifold M (cf. [DrTo]). Extrinsic construction of Fefferman's metric $g \in \mathrm{Lor}(\partial\Omega \times S^1)$ for a smoothly bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ (cf. [Fef]). Graham's connection 1-form (cf. [Gra]) and the intrinsic construction of Fefferman's metric $F_\theta \in \mathrm{Lor}(C(M))$ on the total space $C(M)$ of the canonical S^1 bundle over a strictly pseudoconvex CR manifold M, equipped with a positively oriented contact form θ (cf. [Lee]). Cartan chains on strictly pseudoconvex real hypersurfaces $M \subset \mathbb{C}^2$ (cf. [Car], [ChMo]). For any strictly pseudoconvex real hypersurface $M \subset \mathbb{C}^n$ the non vertical null geodesics of the Fefferman metric on $M \times S^1$ project on Chern-Moser-Cartan chains (cf. [Fef]). Any two nearby points on a strictly pseudoconvex real hypersurface $M \subset \mathbb{C}^n$ can be joined by a chain (cf. [Jac], [Koc]). The scalar curvature K of $(C(M), F_\theta)$ is S^1 invariant and its pushforward $\pi_* K$ is a constant multiple of the pseudohermitian scalar curvature R of [the Tanaka-Webster connection of] (M, θ) (cf. [Lee]). The geometric wave operator $\square$ [the Laplace-Beltrami operator of the Lorentzian manifold $(C(M), F_\theta)$] is S^1 invariant and $\pi_* \square = \Delta_b$ [the sublaplacian of (M, θ)] (cf. [Lee]). The Pontryagin forms of the Fefferman metric are CR invariants (cf. [BaDr]). The sphere $M = S^{2n-1} \subset \mathbb{C}^n$ has a vanishing Pontryagin form $P_1(\Omega) = 0$ and the corresponding transgression class has integer coefficients (cf. [BaDr]). The base map $\phi : M \to N$ corresponding to any S^1 invariant harmonic map $\Phi : C(M) \to N$ [of the Lorentzian manifold $(C(M), F_\theta)$ into the Riemannian manifold N] is a subelliptic harmonic map (cf. [BDU], [JoXu]).

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