Lecture 4

Singularities of Space-Times, Schmidt's Bundle Boundaries, Clarke's Singular Holonomy Groups, Quantum Mechanical "healing" of Space-Time Singularities[1]

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Singularities of space-times, Albert Einstein's accessibility criterium (cf. [Ein]). Bundle boundary constructions: Schmidt's metric G_b on the total space $O(\mathscr{M}, g)$ of the principal O(1, 3)-bundle of Lorentz frames tangent to the space-time $(\mathscr{M}, g)$. The distance function $d_{G_b} : O^+(\mathscr{M}, g) \times O^+(\mathscr{M}, g) \to [0, +\infty)$ [$O^+(\mathscr{M}, g)$ = connected component of $O(\mathscr{M}, g)$] associated to G_b as a Riemannian metric. The right translations $R_a : O^+(\mathscr{M}, g) \to O^+(\mathscr{M}, g)$ are uniformly continuous with respect to d_{G_b}. The proper ortho-crone Lorentz group $\mathscr{L}^\uparrow = \mathrm{O}^{++}(1, 3)$ acts on $\overline{O^+(\mathscr{M}, g)}$ [the Cauchy completion of $O(\mathscr{M}, g)$ with respect to d_{G_b}] as a topological transformation group, and $\mathscr{M} \simeq O^+(\mathscr{M}, g)/\mathscr{L}^\uparrow$ embeds into $\overline{\mathscr{M}} := \overline{O^+(\mathscr{M}, g)}/\mathscr{L}^\uparrow$. Schmidt's bundle boundary $\partial_b \mathscr{M} = \overline{\mathscr{M}} \setminus \mathscr{M}$. Characterization of bundle boundary points (as end points $\lim_{t\to 1^-} \alpha(t)$ of inextensible curves $\alpha : [0, 1) \to \overline{\mathscr{M}}$ of finite b-length, cf. [Sch1]). Each fibre of $\Pi_{\overline{O}} : \overline{O^+(\mathscr{M}, g)} \to \overline{\mathscr{M}}$ over a bundle boundary point is (cf. [Joh1]) isomorphic to a homogeneous space of the Lorentz group

$$\Pi_{\overline{O}}^{-1}(p) \approx \mathscr{L}_+^\uparrow / G_p\,, \quad p \in \partial_b \mathscr{M}\,,$$

for some subgroup $G_p \subset \mathscr{L}_+^\uparrow$, which is Clarke's singular holonomy group at p. Construction of G_p (cf. [Cla]). G_p consists of the Lorentz transformations that can be generated by parallel displacement along "arbitrarily short" loops near p (cf. [Cla]). G_p contains a subgroup $\tilde{G}_p$ which is completely

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determined by the Riemann curvature tensor of $(\mathscr{M},\, g)$. If p is a curvature singularity (cf. [Tho]) then generically $G_p = \tilde{G}_p = \mathscr{L}_+^{\uparrow}$ and the fibre $\Pi_{\overline{O}}^{-1}(p)$ is a single point. G.T. Horowitz & D. Marolf's alternative manner (cf. [HoMa]), quantum mechanically based, of inspecting classical singularities of space-times. Motion of a quantum test particle in a classically singular space-time. For any static space-time $(\mathscr{M},\, g)$, and any time-like Killing vector field ξ [locally represented as $\xi = \partial/\partial t$], the decomposition of the wave equation

$$\big(\nabla^\mu \nabla_\mu - m^2\big)\Psi = 0$$

with respect to the direct sum decomposition $T_p(\mathscr{M}) = T_p(\Sigma) \oplus \mathbb{R}\xi_p$ for any $p \in \Sigma$ [Σ = hypersurface of $\mathscr{M}$ orthogonal to ξ] i.e.

$$\frac{\partial^2 \Psi}{\partial t^2} = -V^2\, \Delta_\Sigma \Psi - V^2 m^2 \Psi$$

with $V := \big[-g\big(\xi,\, \xi\big)\big]^{1/2}$ (the potential function). The operator

$$A\Psi \equiv V^2\, \Delta_\Sigma \Psi + V^2 m^2 \Psi$$

[(minus) the spacial part of the Klein-Gordon operator] thought of as an operator of Hilbert spaces

$$A : \mathscr{D}(A) \subset L^2\big(\Sigma,\, d\mu\big) \to L^2\big(\Sigma,\, d\mu\big), \quad d\mu = \frac{1}{V}\, d\,\mathrm{vol}(g_\Sigma).$$

A is symmetric in the sense of Lagrange i.e.

$$A_0 \equiv A\big|_{C_0^\infty(\Sigma)} : C_0^\infty(\Sigma) \subset L^2(\Sigma,\, d\mu) \to L^2(\Sigma,\, d\mu)$$

is a symmetric operator. A_0 is a real operator. A_0 admits self adjoint extensions. G.T. Horowitz and D. Marolf's key question "is such an extension is unique?" (cf. [HoMa]). The case where A_0 admits a unique selfadjoint extension A_E (cf. [HoMa]). A_E coincides with the Friedrichs extension $\tilde{A}_0$. A_E is positive. The positive selfadjoint square root $\sqrt{A_E}$ of A_E is well defined (cf. [Ber], [SeTa]). The wave equation $\partial^2\Psi/\partial t^2 = -A_0\Psi$ is (the wave function for a free relativistic particle satisfies)

$$i\, \frac{\partial \Psi}{\partial t} = \sqrt{A_E}\, \Psi$$

with the solution $\Psi(t) = \exp\big[-i\, t\, \sqrt{A_E}\big]\, \Psi(0)$ (the evolution of any state is uniquely defined for all time, by an initial condition). Quantum singular space-times (A_0 is not essentially selfadjoint, cf. [HoMa]).

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