Lecture 5

Curvature Singularities of Pseudoconvex Boundaries, Gravitational Field Equations on Fefferman spaces, Wave Maps from Gödel's Universe[1]

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Diederich-Fornaess' "worm" domains $\Omega = \{(z, w) \in \mathbb{C}^2 \, : \, \rho(z, w) < 0\}$ $\big(\rho(z, w) := \big|z - \exp(i \log |w|^2)\big|^2 - 1 + \eta\big(\log |w|^2\big)$, cf. [DiFo]$\big)$. Failure of Condition R for the Bergman projection

$$Pf(z, w) = \int_\Omega K(z, w, \zeta, \omega)\, f(\zeta, \omega)\, dV(\zeta, \omega), \quad (\zeta, \omega) \in \Omega\,,$$

(P doesn't map $C^\infty(\Omega)$ into itself, cf. [Chr]). The Levi flat locus of the boundary $\partial\Omega$ is the annulus $\mathscr{A} = \{(0, w) \in \partial\Omega \, : \, |\log |w|^2| \leq \mu\}$ and $M = \partial\Omega \setminus \mathscr{A}$ is a strictly pseudoconvex CR manifold. Structure circles

$$\gamma_{w_0}(\varphi) = \big(2\,\cos\big[\varphi - \log |w_0|^2\big]\, e^{i\varphi}\,,\, w_0\big) \in M\,, \quad \varphi \in I(w_0)\,,$$

$$I(w_0) = \Big[\log |w_0|^2\,,\, \log |w_0|^2 + \frac{\pi}{2}\Big) \subset \mathbb{R}, \quad w_0 \in \mathbb{C} \setminus \{0\}\,, \quad \big|\log |w_0|^2\big| \leq \mu\,,$$

wind at a Levi flat point $(0, w_0) \in \mathscr{A}$. Every lift $\Gamma(\varphi) \in C(M)$ runs into a curvature singularity of the Fefferman metric $F_\theta \in \mathrm{Lor}(C(M))$ (with $\theta = \frac{i}{2}\,\mathbf{j}^*\big(\overline{\partial} - \partial\big)\rho$, $\mathbf{j} : \partial\Omega \hookrightarrow \mathbb{C}^2$, cf. [BDP]).

The total space $\mathscr{M} \simeq \mathbb{H}_1 \times S^1$ of the canonical circle bundle over the 3-dimensional Heisenberg group $\mathbb{H}_1$ is a space-time with Fefferman's metric F_{θ_0} associated to the canonical Tanaka-Webster flat contact form $\theta_0 = dt + i\big(z\, d\overline{z} - \overline{z}\, dz\big)$. F_{θ_0} isn't flat. The matter and energy content of $(\mathscr{M}, F_{\theta_0})$ is described by the energy momentum tensor $\mathring{T}_{\mu\nu}$ [the trace-less Ricci tensor of F_{θ_0}]. First order perturbation $g = F_{\theta_0} + \epsilon\, h$, $\epsilon << 1$, and linearization of the gravitational field equations $R_{\mu\nu} - \frac{1}{2}\, R\, g_{\mu\nu} = \mathring{T}_{\mu\nu}$ on $\mathscr{M}$ (cf. [BDJ1]).

Gödel's universe $\mathcal{G}^4_\alpha = (\mathbb{R}^4, g_\alpha)$ realized as the total space of a principal $\mathbb{R}$-bundle over a strictly pseudoconvex CR manifold M^3 (cf. [Koc]). For any $\mathbb{R}$-invariant wave map Φ of $\mathcal{G}^4_\alpha$ into a Riemannian manifold N,

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the corresponding base map $\phi : M^3 \to N$ is subelliptic harmonic, with respect to a canonical choice of contact form θ on M^3. The subelliptic Jacobi operator J_b^ϕ of ϕ has a discrete Dirichlet spectrum (cf. [BDM]) on any bounded domain $D \subset M^3$ supporting the Poincaré inequality on $\mathring{W}_H^{1,2}(D, \phi^{-1}TN)$ and Kondrakov compactness [compactness of the embedding $\mathring{W}_H^{1,2}(D, \phi^{-1}TN) \hookrightarrow L^2(D, \phi^{-1}TN)$]. The solution $\pi : \mathcal{G}_\alpha^4 \to M^3$ to the wave map system on $\mathcal{G}_\alpha^4$ has index $\mathrm{ind}^\Omega(\pi) \geq 1$, for any bounded domain $\Omega \subset \mathcal{G}_\alpha^4$. Mounoud's distance $d^\infty_{G_0,\Omega}(g_\alpha, F_\theta)$ is bounded below by a constant depending only on the rotation frequency of Gödel's universe [a measure of the bias of g_α from being Fefferman-like in the region $\Omega \subset \mathbb{R}^4$].

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