Curvature Structure and Metric in General Relativity Theory:a brief preface.

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These two lectures describe a $4-$dimensional manifold M admitting a Lorentz metric g, and collectively labelled (M, g). This is the space-time of general relativity, but these lectures are really about $4-$dimensional Lorentzian geometry.

LECTURE 1. The Curvature and Weyl Tensors.

In the first lecture some elementary $4-$dimensional tangent space geometry is described for M and then the Riemann curvature tensor *Riem* is introduced. The metric g leads to a unique Levi-Civita connection, ∇ and this, in turn, uniquely determines the curvature tensor *Riem* (with coordinate components $R^a{}_{bcd}$). The question then arises: to what extent does the specification of the curvature tensor (known, say, to arise from some metric) determine that metric? This problem will be addressed and to facilitate it a classification of *Riem* will be described. This is done using a decomposition of *Riem* into its bivector structure.

This enables one to describe a similar problem regarding the *Weyl conformal tensor* C on M with coordinate components $C^a{}_{bcd}$. Such a tensor requires a metric for its definition (unlike *Riem* which needs only a connection). If two metrics g and g' on M satisfy $g' = \phi g$ for a smooth real-valued function ϕ on M, they are said to be conformally related and Weyl's theorem shows that their Weyl conformal tensors are equal. One then asks: does the converse of this theorem hold? This problem will be solved here. To do this one decomposes C into its bivector structure (similar to the technique used by Petrov in his famous classification of C). A reference for the ideas in lecture 1 is the present author's text "Symmetries and Curvature Structure in General Relativity" World Scientific, 2004. A more detailed bibliography will be given in the associated lectures notes.

LECTURE 2. Sectional Curvature and Geodesic Structure.

In this lecture, the *sectional curvature* function on M is brought into play. This is determined by the metric g and consists at each $m \in M$ of the collection of all Gauss curvatures of all non-null $2-$spaces "starting from m". This is reminiscent of Riemann's original conception of curvature. Again one asks is the converse of this theorem true, that is, does the sectional curvature function on M determine the metric from whence it came? This lecture also answers the question of whether it makes sense to ask about the sectional curvature of a null $2-$space.

Also in this last lecture one can imagine the collection of all geodesic paths on M for some Levi-Civita connection ∇ on M arising from a metric g and ask the question as to what extent this collection of geodesic paths fixes the connection and the metric. This allows us to introduce the *Weyl projective tensor* W which plays a similar role here to that played by C in the previous lecture.

These lectures raise the question as to what is the most convenient choice of gravitational potential in general relativity theory. A reference for the ideas in lecture 2 is "Symmetries and Curvature Structure in General Relativity" World Scientific, 2004. A more detailed bibliography will be given in the associated lectures notes.